# Unit 10
## Visualizing Geometric Relationships

### Lesson Outline

### BIG PICTURE

Students will:
- develop geometric relationships involving right-angled triangles, and solve problems involving right-angled triangles geometrically.

<table>
<thead>
<tr>
<th>Day</th>
<th>Lesson Title</th>
<th>Math Learning Goals</th>
<th>Expectations</th>
</tr>
</thead>
</table>
| 1   | Building Squares | • Activate prior knowledge.  
• Explore the Pythagorean relationship, using manipulatives (tangrams, chart paper). | 8m49  
CGE 3c, 5a |
| 2   | Squared From All Sides  
*GSP®4 file: Pythagorean Puzzle* | • Explore and investigate, using concrete materials, the relationship between the area of the squares on the legs and the area of the square on the hypotenuse of a right-angled triangle. | 8m49  
CGE 3c, 4f, 5e |
| 3   | Challenges Are Shaping Up…  
*GSP®4 file: N-agon Areas* | • Investigate the relationship of the areas of semi-circles drawn on the sides of a right-angled triangle. | 8m49  
CGE 5g, 7b |
| 4   | Pythagoras In Proportion | • Solve problems involving right-angled triangles geometrically, using the Pythagorean relationship and proportionality.  
• Hypothesize and investigate the relationship between the area of similar figures drawn on the sides of a right-angled triangle. | 8m50  
CGE 5b |
| 5   | Instructional Jazz | | |
| 6   | Mathematics in Early Greece  
*(lesson not included)* | • Investigate the definition and historical study of polyhedra.  
• Construct the five Platonic solids.  
http://matti.usu.edu/nlvm/nav/vlibrary.html  
Index → Platonic Solids → Geometry (6–8) | 8m51  
CGE 7f |
| 7   | What’s the Connection?  
*(lesson not included)* | • Record and organize data consisting of the number of faces, vertices, and edges for each platonic solid.  
• Conjecture a possible relationship between the number of faces, vertices, and edges of a polyhedra. | 8m51, 8m61, 8m68, 8m70, 8m73, 8m78  
CGE 3c, 5b |
| 8   | Impossible Shapes  
*(lesson not included)* | • Test the hypothesis from Day 2 by constructing and examining non-Platonic solids.  
• Using the relationship formula developed, investigate impossible polyhedra shapes. | 8m62, 8m78  
CGE 3b, 3c |
| 9   | Instructional Jazz | | |
| 10  | Please Move  
*(lesson not included)* | • Investigate and report on real-world examples of translations, reflections, and rotations. | 8m53  
CGE 5b, 7f |
| 11  | Shifty Business  
*(lesson not included)* | • Translate single points and sets of points horizontally, vertically, and through a combination of both directions.  
• Identify how the type of transformation affects the original point’s coordinates. | 8m52  
CGE 3c, 4b, 5a |
<table>
<thead>
<tr>
<th>Day</th>
<th>Lesson Title</th>
<th>Math Learning Goals</th>
<th>Expectations</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Points to Reflect Upon</td>
<td>• Reflect single points and sets of points in the x-axis, and in the y-axis.</td>
<td>8m52 CGE 3c, 4b, 5a</td>
</tr>
<tr>
<td></td>
<td><em>(lesson not included)</em></td>
<td>• Identify how the type of transformation affects the original point’s coordinates.</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>A New Slant on Reflection</td>
<td>• Reflect single points and sets of points in the line that forms the angle bisector of the x- and y-axes and passes through the first and third quadrants.</td>
<td>8m52 CGE 3c, 4b, 5a</td>
</tr>
<tr>
<td></td>
<td><em>(lesson not included)</em></td>
<td>• Identify how the type of transformation affects the original point’s coordinates.</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Getting Dizzy</td>
<td>• Rotate single points and sets of points through 90, 180, and 270 degrees about the origin.</td>
<td>8m52 CGE 3c, 4b, 5a</td>
</tr>
<tr>
<td></td>
<td><em>(lesson not included)</em></td>
<td>• Identify how the type of transformation affects the original point’s coordinates.</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Summative Assessment</td>
<td></td>
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</tr>
</tbody>
</table>
Unit 10: Day 1: Building Squares

Math Learning Goals
• Activate prior knowledge.
• Explore the Pythagorean relationship, using manipulatives (tangrams, chart paper).

Materials
• 3 sets of tangram pairs
• chart paper
• BLM 10.1.1, 10.1.2

Assessment Opportunities
Use paper cut-outs of tangram pieces as an alternative.

Word Wall
• leg
• hypotenuse
• vertices
• right angle

Minds On…
Whole Class → Brainstorm
List examples of, or references to, the use of right-angled triangles, e.g., bridges, trusses, flags.
Ask: How do we identify a right-angled triangle? Students use specific vocabulary in responding: e.g., legs, hypotenuse, vertices, right angle.

Action!
Pairs → Investigation
Explain the task:
Select one large triangular tangram piece and identify it as the central piece. Using the remaining tangram pieces, construct 3 perfect squares off the legs and hypotenuse. Trace each piece onto blank paper.

Determine if there is a relationship between the pieces of the tangram on the legs and the pieces on the hypotenuse.
Post students’ work.

Learning Skills/Initiative/Rating Scale: Observe how students interact with their peers.

Consolidate Debrief
Whole Class → Discussion
Students explain their solutions.
Ask:
• What did you discover?
• Which pieces did you use?
• How are the pieces related?

Determine how the pieces of the square on the legs fit onto the pieces of the square on the hypotenuse.
Ask: What does this tell you about right-angled triangles?

Home Activity or Further Classroom Consolidation
• Create your own piece-puzzles, using worksheet 10.1.1. Make a right-angled triangle and construct squares on the legs and hypotenuse of it. Cut the squares on the legs into pieces so that they fit exactly onto the square on the hypotenuse.
• Research two interesting notes about Pythagoras’ life, unrelated to mathematics.

See BLM 10.1.2 for solutions.
Students’ research is due for Day 5.
10.1.1: Tangram Squares
10.1.2: Tangram Squares (Solutions)

Solution 1

Tangrams on the legs fit on the square on the hypotenuse.
10.1.2: Tangram Squares (Solutions)

Solution 2
Math Learning Goals
- Explore and investigate, using concrete materials, the relationship between the area of the squares on the legs and the area of the square on the hypotenuse of a right-angled triangle.

Minds On… Whole Class → Discussion
Review students’ solutions to their Home Activity puzzles. Use an available representation, e.g., chart to be left up for display, poster, overhead representation using transparent tangram, Pythagorean Puzzle GSP®4. Connect results to work with tangrams on Day 1. Ask: Is there a pattern?

Individual → Journal
Draw the standard diagram for illustrating the relationship of the squares drawn on the sides of a right-angled triangle, identifying legs and hypotenuse titled Pythagorean Relationship.

Think/Pair/Share → Anticipation Guide
Students highlight key words, then complete the Before column on the Anticipation Guide (BLM 10.2.1) and explain their reasoning to a partner.

Action! Pairs → Investigation
Pairs investigate which combinations of squares will successfully form a right-angled triangle and which will not form a right-angled triangle.

Students cut out 12 different squares and arrange them in groups of three such that the side lengths create triangles. Using graph paper, they determine if the triangle is a right-angled triangle. They glue down the squares and create their own chart, using BLM 10.2.2.

Reasoning & Proving/Observation/Checklist: Observe how students talk about and record their thinking during the investigation.

Consolidate Debrief
Whole Class → Summarizing
Consolidate the investigation by completing a class summary chart.

Summarize the Pythagorean relationship: In a right-angled triangle, the sum of the areas of the squares on the legs equals the area of the square on the hypotenuse.

Students copy the Pythagorean relationship into the summary (BLM 10.2.3).

Students complete the After column on the Anticipation Guide for questions 1–3.

Home Activity or Further Classroom Consolidation
Complete worksheet 10.2.4.
### 10.2.1: Anticipation Guide for Right-Angled Triangles

**Instructions:**
- Check Agree or Disagree beside each statement in the *Before* column.
- Compare your choice and explanation with a partner.
- Revisit your choices at the end of the task.
- Check Agree or Disagree beside each statement in the *After* column.
- Compare the choices that you would make after the task with the choices that you made before the task.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Before Agree</th>
<th>Before Disagree</th>
<th>After Agree</th>
<th>After Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Right-angled triangles can sometimes be isosceles.</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>2. There is a special relationship between the squares of the three sides of a right-angled triangle.</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. There is a special relationship between the squares of the sides for any triangle.</td>
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<tr>
<td>4. There is a special relationship between the areas of similar shapes that fit on the sides of a right-angled triangle.</td>
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<td></td>
</tr>
</tbody>
</table>
10.2.2: Pythagorean Puzzle

Cut out the squares.
Explore which combinations of squares will successfully form a right-angled triangle.
Use graph paper to make sure your right angle is exactly $90^\circ$.
Glue down the squares.
**10.2.2: Pythagorean Puzzle (continued)**

Cut out the squares.
Explore which combinations of squares will NOT form a right-angled triangle.
Use grid (graph) paper to make sure your angle is NOT 90°.
Glue down the squares.
10.2.3: Squared From All Sides Summary Chart

Name:

**Part A: Squares that form a right-angled triangle**

<table>
<thead>
<tr>
<th>Triangle #</th>
<th>Label of Square on Leg 1</th>
<th>Area of Square on Leg 1</th>
<th>Label of Square on Leg 2</th>
<th>Area of Square on Leg 2</th>
<th>Label of Square on Hypotenuse</th>
<th>Area of Square on Hypotenuse</th>
<th>Area Part C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
</tbody>
</table>

**Part B: Squares that do not form a right-angled triangle**

<table>
<thead>
<tr>
<th>Triangle #</th>
<th>Label of Square on Leg 1</th>
<th>Area of Square on Leg 1</th>
<th>Label of Square on Leg 2</th>
<th>Area of Square on Leg 2</th>
<th>Label of Square on Hypotenuse</th>
<th>Area of Square on Hypotenuse</th>
<th>Area Part C</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>

**Part C: Investigate**
Add the area of the square on Leg 1 to the area of the square on Leg 2. What pattern do you notice?

**Summary**
Pythagoras was a famous Greek philosopher, Olympic coach, and mathematician. He was born on the island of Samos sometime in the sixth century B.C.E. He is credited with discovering the Pythagorean relationship, which states:
10.2.4: Follow-Up Chart for Pythagorean Relationship Investigation

Fill in the blanks on the chart.

<table>
<thead>
<tr>
<th>Right-Angled Triangle</th>
<th>Area of Square on Leg 1</th>
<th>Area of Square on Leg 2</th>
<th>Area of Square on Hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABC</td>
<td>9 cm²</td>
<td>16 m²</td>
<td></td>
</tr>
<tr>
<td>DEF</td>
<td></td>
<td>15 mm²</td>
<td>64 mm²</td>
</tr>
<tr>
<td>STU</td>
<td>121 cm²</td>
<td>36 cm²</td>
<td></td>
</tr>
<tr>
<td>XYZ</td>
<td>100 cm²</td>
<td>30.25 cm²</td>
<td></td>
</tr>
<tr>
<td>LMN</td>
<td>16 cm²</td>
<td></td>
<td>100 cm²</td>
</tr>
</tbody>
</table>
**Pythagorean Puzzle** (GSP®4 file)

**Pythagorean Puzzle.gsp**

**Pythagorean Puzzle**

*Given: Right Triangle ABC, \( \angle ACB = 90^\circ \).

Squares are drawn on the three sides and different sections are colored.

Move the sections from the smaller squares and fit them into the large square at the bottom.

What property of right triangles do you think is illustrated here?

---

**The Pythagorean Relationship**

A right angled triangle is shown with a right angle at C.

- Show Pythagorean Relationship
- Reset

Drag this point
Pythagorean Relationship (GSP®4 file continued)

The Pythagorean Relationship

Given: Right Triangle ABC, $\angle ACB = 90^\circ$.
Squares will be appear on each of the three sides when you click on the button:

- Show Objects
- Hide Objects

Click in the colored section of each square then under Measure choose Area.

Under the Measure tab choose Calculate and determine the sum of the areas of AEFC and CGHB.

What do you notice about the sum of these two areas? ______

The Pythagorean Relationship

A right angled triangle is shown with a right angle at A.

Follow the steps below.

1) Show Squares of Sides
2) Show Altitude
3) Show Quadrilaterals
4) Show Area Measurements

...
Math Learning Goals
- Investigate the relationship of the areas of semi-circles drawn on the sides of a right-angled triangle.

Minds On...
Whole Class → Discussion
Collect the Home Activity for assessment.
Using a sketch, reinforce the concept of the Pythagorean relationship. Stress that the relationship is true for right-angled triangles only.
Ask: Does this relationship work with shapes other than squares drawn on the right sides of a right-angled triangle?

Action!
Pair/Share → Investigation
Using grid paper, students draw a right-angled triangle. They construct semi-circles on the legs and hypotenuse of the triangle and calculate the areas of each semi-circle to determine the relationship the same way they did with squares on Day 2. Students share their work with another pair and explain their reasoning.
Reasoning & Proving/Observation/Checklist: Observe students as they explain their reasoning.

Consolidate Debrief
Whole Class → Discussion/Brainstorm
Summarize the findings of their investigation. The sum of the area of the semi-circles on the legs is equal to the area of the semi-circle on the hypotenuse. Pythagorean relationship works for a right-angled triangle using squares and semi-circles drawn on the sides.
Ask:
- What other shapes will work?
- Under what conditions will other shapes work?
Students complete the After column for question 4 of the Anticipation Guide (Day 2 BLM 10.2.1).

Home Activity or Further Classroom Consolidation
Draw a right-angled triangle with the length of legs being whole numbers. On each side of the triangle draw a rectangle (no squares are allowed!). Calculate the areas of the three rectangles. Does this demonstrate the Pythagorean relationship? Explain. Repeat with two more triangles.
N-agon Areas (GSP®4 file)
N-agon Areas.gsp

Green Area = 15.10 cm$^2$
Yellow Area = 10.45 cm$^2$
Blue Area = 4.64 cm$^2$
(Yellow Area)+(Blue Area) = 15.10 cm$^2$

<table>
<thead>
<tr>
<th>Green Area</th>
<th>(Yellow Area)+(Blue Area)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.91 cm$^2$</td>
<td>3.91 cm$^2$</td>
</tr>
<tr>
<td>7.21 cm$^2$</td>
<td>7.21 cm$^2$</td>
</tr>
<tr>
<td>44.16 cm$^2$</td>
<td>44.16 cm$^2$</td>
</tr>
<tr>
<td>24.95 cm$^2$</td>
<td>24.95 cm$^2$</td>
</tr>
<tr>
<td>15.10 cm$^2$</td>
<td>15.10 cm$^2$</td>
</tr>
</tbody>
</table>

Double click here to change the number of sides on the polygon = 4.00

Green Area = 9.88 cm$^2$
Blue Area = 8.63 cm$^2$
Yellow Area = 1.25 cm$^2$
(Blue Area)+(Yellow Area) = 9.88 cm$^2$

<table>
<thead>
<tr>
<th>Green Area</th>
<th>(Blue Area)+(Yellow Area)</th>
</tr>
</thead>
<tbody>
<tr>
<td>72.42 cm$^2$</td>
<td>72.42 cm$^2$</td>
</tr>
<tr>
<td>76.50 cm$^2$</td>
<td>76.50 cm$^2$</td>
</tr>
<tr>
<td>35.90 cm$^2$</td>
<td>35.90 cm$^2$</td>
</tr>
<tr>
<td>9.88 cm$^2$</td>
<td>9.88 cm$^2$</td>
</tr>
</tbody>
</table>
N-agon Areas (GSP®4 file continued)

Double click here to change the number of sides on the polygon = 6.00

Green Area = 25.67 cm²
Blue Area = 22.41 cm²
Yellow Area = 3.25 cm²

(Blue Area)+(Yellow Area) = 25.67 cm²

<table>
<thead>
<tr>
<th>Green Area</th>
<th>(Blue Area)+(Yellow Area)</th>
</tr>
</thead>
<tbody>
<tr>
<td>72.42 cm²</td>
<td>72.42 cm²</td>
</tr>
<tr>
<td>76.50 cm²</td>
<td>76.50 cm²</td>
</tr>
<tr>
<td>35.90 cm²</td>
<td>35.90 cm²</td>
</tr>
<tr>
<td>25.67 cm²</td>
<td>25.67 cm²</td>
</tr>
</tbody>
</table>
**Math Learning Goals**

- Solve problems involving right-angled triangles geometrically, using the Pythagorean relationship and proportionality.
- Hypothesize and investigate the relationship between the areas of similar figures drawn on the sides of a right-angled triangle.

**Minds On… Whole Class → Discussion**

Students share the results of their Home Activity. Volunteers record their area measurements and sketches on the board. Discuss why their investigation did not show a Pythagorean relationship. Why do squares and semi-circles work? Stress similar shapes.

**Action! Pairs → Investigation**

Pairs use a GSP® 4 sketch to investigate the hypothesis: Areas of similar figures drawn on the sides of a right-angled triangle show a Pythagorean relationship.

They record observations and patterns, and explain their reasoning.

**Reasoning & Proving/Exploration/Checklist:** Observe as students’ investigate and look for opportunities to probe for generalization of the relationship.

**Consolidate Debrief**

**Small Group → Reflection**

Identify which type of polygon can be used on the sides of a right-angled triangle to create the Pythagorean relationship. Guide students to discover that only similar polygons fulfill the relationship.

Students include one of the GSP® 4 sketches they investigated, along with a general statement about the Pythagorean relationship and similar polygons.

Create a class Frayer Model on the Pythagorean relationship. Post titles in four different locations of the room: Definition, Facts/Characteristics, Examples, Non-examples. Working in small groups, students respond at each station, phrasing or rephrasing and adding to the previous group’s work. Assemble a large poster to display as a Frayer Model (BLM 10.4.2).

**Home Activity or Further Classroom Consolidation**

Complete worksheet 10.4.1.
### 10.4.1: Test a Triangle – Is It a Right-Angled Triangle? (Teacher)

Test each of the following triangles and determine if the triangle is a right-angled triangle:

<table>
<thead>
<tr>
<th>Side 1</th>
<th>Side 2</th>
<th>Longest side</th>
<th>Areas</th>
<th>Is this a right-angled triangle?</th>
<th>How do you know?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>8</td>
<td>15</td>
<td>17</td>
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<td>7</td>
<td>10</td>
<td>13</td>
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<td>8</td>
<td>12</td>
<td>15</td>
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<tr>
<td>9</td>
<td>40</td>
<td>41</td>
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</tr>
</tbody>
</table>
### 10.4.1: Test a Triangle – Is It a Right-Angled Triangle? (continued)

<table>
<thead>
<tr>
<th>Side 1</th>
<th>Side 2</th>
<th>Longest side</th>
<th>Areas</th>
<th>Is this a right-angled triangle?</th>
<th>How do you know?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>(3^2 + 4^2) = 25</td>
<td>yes</td>
<td>(3^2 + 4^2 = 5^2)</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>7</td>
<td>(4^2 + 6^2) = 52</td>
<td>no</td>
<td>(4^2 + 6^2 \neq 7^2)</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>13</td>
<td>(5^2 + 12^2) = 169</td>
<td>yes</td>
<td>(5^2 + 12^2 = 13^2)</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>17</td>
<td>(8^2 + 15^2) = 289</td>
<td>yes</td>
<td>(8^2 + 15^2 = 17^2)</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>13</td>
<td>(7^2 + 10^2) = 149</td>
<td>no</td>
<td>(7^2 + 10^2 \neq 13^2)</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>15</td>
<td>(8^2 + 12^2) = 208</td>
<td>no</td>
<td>(8^2 + 12^2 \neq 15^2)</td>
</tr>
<tr>
<td>9</td>
<td>40</td>
<td>41</td>
<td>(9^2 + 40^2) = 1681</td>
<td>yes</td>
<td>(9^2 + 40^2 = 41^2)</td>
</tr>
</tbody>
</table>