The Literacy and Numeracy Secretariat

CAPACITY BUILDING SERIES

SECRETARIAT
SPECIAL EDITION #7

The Capacity Building Series is produced by The Literacy and Numeracy Secretariat to support leadership and instructional effectiveness in Ontario schools. The series is posted at: www.edu.gov.on.ca/eng/literacynumeracy/inspire/ For information: lns@ontario.ca

Why Differentiation?

"Differentiation is an organized yet flexible way of proactively adjusting teaching and learning to meet kids where they are and help them to achieve maximum growth as learners." (Tomlinson, 1999)

September 2008
ISSN: 1913 8482 (Print)
ISSN: 1913 8490 (Online)

Differentiating Mathematics Instruction

The purpose of differentiating instruction in all subject areas is to engage students in instruction and learning in the classroom. All students need sufficient time and a variety of problem-solving contexts to use concepts, procedures and strategies and to develop and consolidate their understanding. When teachers are aware of their students' prior knowledge and experiences, they can consider the different ways that students learn without pre-defining their capacity for learning.

Teachers can help students achieve their potential as learners by providing learning and consolidation tasks that are within the student's "zone of proximal development." The zone of proximal development, a phrase coined by the psychologist, Lev Vygotsky, refers to the student's capacity for learning. Technically, it is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Ontario Ministry of Education, 2005a, p. 61). Identifying the student's zone of proximal development is of paramount importance if differentiated instruction is to achieve its maximum impact.

This monograph focuses on differentiating instruction in the mathematics classroom. It describes several classroom strategies for differentiating mathematics instruction – namely, focusing instruction on key concepts, using an instructional trajectory or learning landscape for planning and designing open and parallel tasks.

Valuing the Diversity of Student Thinking

Students in any classroom differ in many ways. When teachers develop instructional plans that acknowledge their differences, students can learn in ways that are suitable for and meaningful to them. Which differences warrant focus for differentiating instruction and assessment? Some differences could be cognitive (e.g., mathematics knowledge, skills, strategies), while other differences could be affective and behavioural (e.g., curiosity, confidence, perseverance).

Differentiating in General

There is a body of work (e.g., Gregory, 2003; Tomlinson,1999, 2001; Tomlinson & McTighe, 2006) that articulates strategies for differentiating instruction by content, learning process and/or product to address the needs of students with varied experiences, learning readiness, learning styles, contextual interests, and learning interaction preferences (e.g., small group, pairs, individual). Many of the techniques described are useful, but none are likely to be effective unless we:

- consider the essential purpose of instruction
- gather data using assessment for learning strategies
- understand students' mathematical needs and readiness, individually and collectively.

Differentiating in Mathematics

- The focus of instruction must be on the key mathematics concepts (big ideas) being taught.
- There must be some aspect of choice for the student, in terms of the details of the learning task, the ways the task can be carried out and how the task is assessed
- Assessment for learning is essential to determine the learning needs of different students.
  (e.g., Dacey & Lynch, 2007; Dacey & Salemi, 2007; Small, in press b)

Imagine this problem being posed in a Grade 3 classroom.

Window Cleaning Problem
Patricia has 3 big windows that need cleaning in her living room.
Each window is made up of 3 rows of 4 panes of glass.
How many panes of glass need to be cleaned?

Take a moment to record a few solutions that you think students might provide. Think about the mathematics knowledge and experiences which students might use to create their solutions.

What different ways could students respond to the problem?

Mary Lou just waits for the teacher to give her more instructions and guidance.
Carmen draws a picture of 1 window in the apartment and counts the panes.
Dane draws a picture of 3 windows and counts the 3 windows not the panes.
Cloyce uses addition and writes 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4.
Barnabas uses addition and writes 4 + 4 + 4 = 12; 12 + 12 + 12 = 36.
Anne uses multiplication and addition 3 x 4 = 12 and 12 + 12 + 12 = 36.

How does a teacher make use of differentiated student responses?

A teacher needs to understand the mathematics represented in the different student responses to the problem and see mathematical connections between the solutions. In this case, students might see the window itself as the object to count rather than the window panes, use repeated addition to show counting of same size groups or use addition and multiplication symbols. Such a range of responses prompts a teacher to recognize that the instructional decisions and interactions with students must be responsive to their mathematical ideas, strategies and communication.

In response to the student solutions to the Grade 3 Window Cleaning Problem, some possible instructional considerations are described below:

Cloyce and Barnabas need to learn the benefits of recording their thinking using simpler symbolism (i.e., multiplication symbol) while recognizing that their initial solutions were appropriate, mathematically.

Anne needs to extend what she already knows to see that she could have recorded the final addition as a multiplication sentence or equation.

Mary Lou needs to be more independent, so the teacher might ask her starting questions like, "What does this problem ask you to do? What two ideas from the problem will you use to make a plan to solve the problem?"

Dane, Mary Lou and Carmen need to solve a problem that is more suitable for their zone of mathematical development, such as "One window has 2 x 6 panes and another window has 3 x 4 panes. Which window has more panes?"

Barnabas needs to see the value of using more sophisticated strategies, so the teacher could set a problem where counting becomes too cumbersome (e.g., larger numbers).

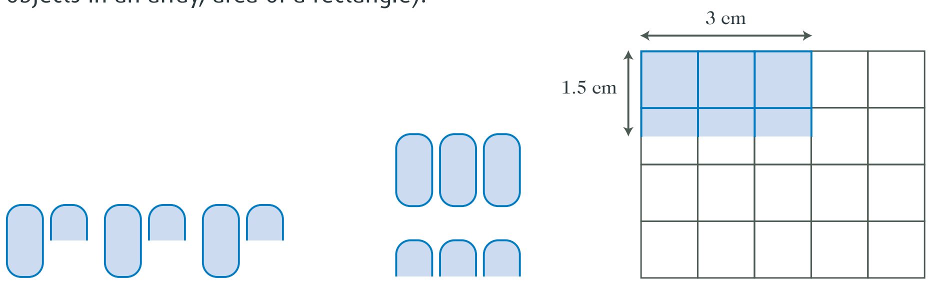
At the same time, the teacher needs to ensure that all students in the class have the opportunity to make a meaningful contribution to the class discussion about their solutions. To co-ordinate a class discussion, the teacher needs to discern the mathematics within and across the student responses, so that sharing of solutions is organized to build collective mathematics knowledge related to the learning goal of the lesson.

1. Focusing Instruction on Key Concepts

Often, the starting point for a lesson can be a narrow learning goal (e.g., one curriculum expectation such as how to carry out a standard multiplication algorithm). It is important to ensure that the required details of the curriculum are considered. But, if a teacher is committed to differentiating mathematics instruction, a better starting point is to consider the *key concepts* that are being addressed.

For example, a Grade 6 teacher might be planning a lesson on multiplying whole numbers by decimals. Although the goal of the instruction is performing a computation like 1.5 x 3, the *key concepts* that students need to understand include:

- Multiplication has many meanings (e.g., repeated addition, counting of equal groups, objects in an array, area of a rectangle).

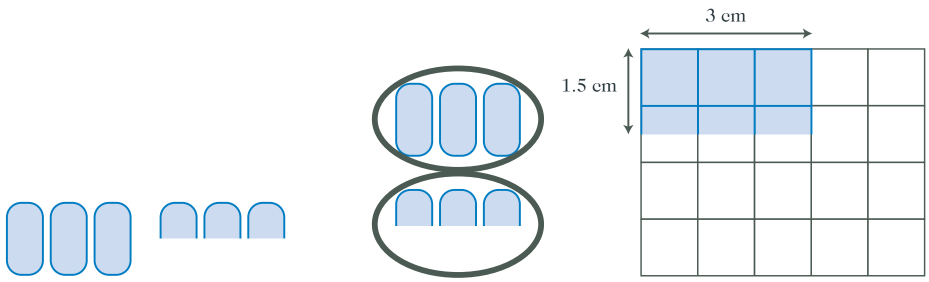


1.5 + 1.5 + 1.5 or 3 groups of 3 and 3 groups of 0.5
3 + 1.5 = 4.5

array of 1.5 rows x 3 columns -> 4.5

1.5 cm x 3 cm = 4.5 cm²

- Multiplication has those same meanings regardless of the values being multiplied.
- Multiplication can be accomplished in parts (distributive property).



1 + 1 + 1 + 0.5 + 0.5 + 0.5 or
3 groups of 1 and 3 groups of 0.5 or
(1 x 3) + (0.5 x 3) = 3 + 1.5 = 4.5

combining 2 arrays of:
1 row x 3 columns and
0.5 row x 3 columns -> 4.5

1 cm x 3 cm + 0.5 cm x 3 cm =
3 cm² + 1.5 cm² = 4.5 cm²

It is impossible to differentiate instruction meaningfully for any mathematics concept, procedure and/or strategy if teachers do not recognize the key concepts (e.g., meanings of multiplication). We could teach the decimal multiplication lesson to students who do not understand decimals (in this case 1.5) by having them explore the concept of decomposing numbers using whole numbers (e.g., 15 x 3 = 10 x 3 + 5 x 3, 3 groups of 10 and 3 groups of 5). But if students do not understand the meanings of multiplication, the lesson will be beyond their capacity; learning and consolidation tasks must meet students in their zones of proximal development.

Overall curriculum expectations are likely to provide a starting point for what the key concepts are, but probably do not help a teacher easily plan to differentiate instruction. One approach is to cluster specific curriculum expectations and use them as learning goals over a series of lessons. It is by clustering specific expectations, in conjunction with looking at the curriculum for other grade levels, that the key concepts become evident. In the Grade 6 multiplication example, the key concepts relate more to what multiplication means, when it is used and its fundamental principles about the operation than the details about what kinds of numbers students can multiply.

Resources for Key Mathematical Concepts

- Charles (2005)
- Ritchhart (1999)
- Schifter, Russell, & Bastable (1999)

For publication details see References and Related Readings on page 8.

An Example of a Specific Cluster of Expectations

- use a variety of mental strategies to solve... multiplication... problems involving whole numbers (e.g. ... use the distributive property...)
- "solve problems involving the multiplication ... of whole numbers, ..."
- "multiply and divide decimal numbers to tenths by whole numbers, ..."

2. Using an Instructional Trajectory/Landscape for Planning

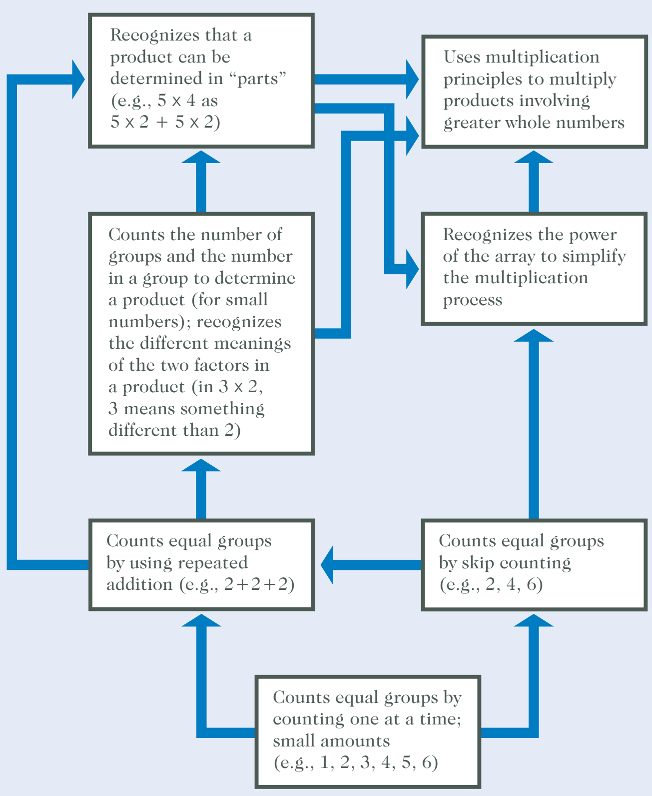
Another critical aspect of meaningful differentiation is considering the student's mathematical development and sophistication. There are many ways to categorize a student's mathematical sophistication, such as stages or phases of cognitive development, developmental continua (Small, 2005; Western Australian Minister of Education and Training, 2006), knowledge packages (Ma, 1999), instructional trajectories (Simon, 1995) and learning landscapes (Fosnot & Dolk, 2001). Although these frameworks have different conceptual emphases, they all describe the development of mathematical learning in a particular topic area. In all of them, what is key is mapping out a sequence of instruction for students.

The Realistic Mathematics Education Group, in the Netherlands, for example, hypothesizes that student mathematical growth can be described in terms of being able to move from familiar contexts to more abstract situations (van den Heuvel-Panhuizen, 2002). Let's consider this idea by thinking through another problem and the different ways that students could respond to it.

> Tables and Seating Problem
>
> At least 79 parents said they are coming to a meeting in our gym tonight.
>
> They will sit at large tables that seat 5 people each.
>
> How many tables do we need?

A Grade 3 student might be able to solve this problem using repeated addition or repeated subtraction, yet not recognize that this problem is an example of a "sharing problem" situation. A more mathematically sophisticated student will see the more general situation; that is, this problem can be solved for any number of parents using the division operation.

Mapping Out a Sequence of Instruction



(Note: Read the trajectory/landscape from the bottom up.)

The organizational frameworks referred to above can be used with clustered specific expectations that represent a key concept to map out the connections among mathematics concepts, strategies and models of representation. Also, such a map can be helpful in recognizing how students' mathematical thinking and understanding could develop across that cluster. Further, when a teacher gathers initial assessment data about students' readiness for learning a particular topic (e.g., beginning of a lesson and a unit of study), the data can be compared to the details of instructional trajectory/landscape to understand students' thinking within the large picture of the key concept.

It is informative to have students respond to an assessment task that is slightly beyond where the teacher hypothesizes students are on the trajectory/landscape. With such a task students could show the full extent of their mathematical thinking in relation to key concepts.

The trajectory/landscape is useful for mapping out a sequence of instruction and for describing where students are in that sequence. Yet, the trajectory/landscape can only be created if the key mathematical concepts are identified and understood in relation to the overall and specific curriculum expectations.

3. Designing Open and Parallel Tasks

Open Tasks

Suppose a Grade 3 teacher wants to approach the key concept that any subtraction can be thought of in terms of a related addition. The Ontario curriculum proposes that Grade 3 students solve addition and subtraction problems involving multi-digit numbers, using concrete materials, student-generated algorithms and standard algorithms, as well as use estimation to help judge the reasonableness of a solution. Some students may not be ready to deal with three-digit numbers, even with the use of concrete materials and personal algorithms. A teacher might change the planned task as follows:

Grade 3 Open Task

There are 316 animal books in the library. 118 of the books are about dogs. The rest are about other animals.

a. How many books are about other animals?
b. How can you add to show that your answer is correct?

revise

Choose a number to use for the missing amount. Solve your problem.

There are ___ animal books in the library. Most of the books are about dogs. The rest are about other animals.

a. How many books are about other animals?
b. How can you add to show that your answer is correct?

With the revised open number task, students have a choice in the numbers they use, choice in the strategies they use and a choice in how they interpret the meaning of the problem. Students who can handle numbers that are below 20 can do so. Students who can handle numbers below 100 in a concrete way can do so. Students who are ready to work with very large numbers can do so. As well, in the revised task, some students will interpret the phrase "most of the books" to mean more than half. Others can simply interpret it as meaning that more books are about dogs than other animals; they might make a list of different animals with a total number of books about each animal, ensuring that the number for dogs is the greatest number on the list.

These variations really don't matter. All of the students will be considering a subtraction situation; all of them are relating it to an addition situation; all of them have an opportunity to understand and solve the problem using their own student-generated strategies and appropriate materials. Whether students are working with large or small numbers, the sharing of their mathematical thinking is valuable for the collective learning of the class.

Great Mathematics Teaching

The open and parallel tasks that you design should be problems that allow students to think mathematically and not just follow learned procedures or routines.

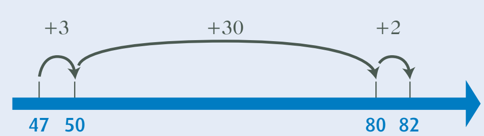
And remember, just like a great movie appeals to a broad audience, great teaching appeals to the learning of a broad spectrum of learners.

Sample Student Solution

I chose 82 books.
Most means more than half.
I found half of 82 by thinking of what number I could add to itself to get 82.

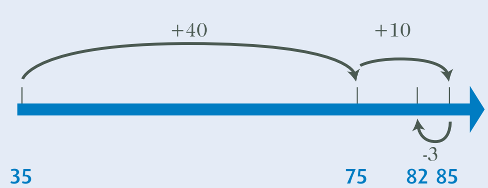
I know that 40 + 40 = 80, so it had to be around 40.
I realized that I just split the extra 2 into 1 + 1, so I know that more than 41 books are about dogs.
I knew I could choose the amount, so I chose 47 books to be about dogs.

To figure out 82 – 47, I added 3 to get to 50 and then 30 to get to 80 and then 2 more.



3 + 30 + 2 = 35 That means there are 35 books about other animals.

I know I'm right since 35 + 47 = 35 + 40 + 7. So 35 + 40 is 75.
75 + 7 is the same as 75 + 10 – 3. That's 85 – 3 = 82.



Let's Make the Most of Our Instructional Time

Meeting each student's needs requires us to ...

- think carefully about the relationship between the mathematical learning goals and the problems chosen
- anticipate what student responses could be and prepare appropriate and strategic follow-up (e.g., co-ordinated class discussion, individual and whole-class questioning, choice of consolidation and practice tasks)

We are not using educational time wisely when ...

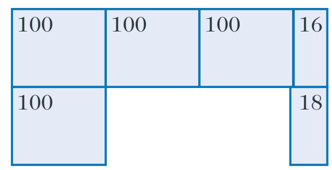
- we are teaching students mathematics they can already deal with independently
- we are assigning tasks that students cannot find solutions for, even with guidance

Types of Open Tasks

- open ended – different strategies, different answers
- open routed – different strategies, same answer

In fact, there might be more mathematically sophisticated thinking from a student who uses a smaller value than one who simply recalls a standard algorithm to subtract 118 from 316. With several differentiated student responses to the problem, it is valuable for students to share their thinking and compare strategies. In this example, the teacher can co-ordinate a class discussion about the use of different models of representations to show different mathematical thinking:

- Some students might use the empty number line. This has the benefit of flexibility; students can use numbers in whatever increments make sense to them.
- Other students might use base ten blocks and focus on place value concepts. These students practise the important skill of decomposing numbers into their hundreds, tens, and ones components.
- Some students might draw diagrams. For example, the student might draw a model for 316 – 118. The model reinforces the mental math concept that to subtract 118 from 316, you can think of subtracting 116 and then another 2, to get 316 – 116 = 200 and 200 – 2 = 198.



Rather than being grade-specific, this use of open tasks is inclusive of all students' mathematical thinking and is relative to students' varied zones of proximal development. In the following example, a Grade 6 teacher might revise a task from her instructional plan by opening it up.

Grade 6 Open Task

How would you determine if a person could be 1 million hours old?

revise

Choose one of these measurements:
- 1000 days,
- 10 000 hours, or
- 1 million seconds

About how old is someone using the measurement you unit chose?

In this example, there are many ways to approach the task, and there is no single correct solution. All students can provide a solution relevant to their personal mathematics knowledge and experience and fully participate in a classroom discussion.

The use of open tasks is in contrast to a more familiar procedure for differentiating mathematics instruction; that is, to break up a task that may be too difficult for some students and ask them to think about a few little bits of the task at a time. This approach, while used for all of the right reasons, reinforces the notion that some students are not capable of independent mathematical thinking and denies some students opportunities to develop that capacity.

Parallel Tasks

Another approach to meeting a variety of needs is to decide on a key concept for learning and to create two parallel tasks that are both focused on that same key concept, yet address students at different levels of mathematical sophistication.

For example, a Grade 3 teacher might wish to help her students see that the difference between two numbers remains the same if the same value is added to both numbers. This principle is equally valid for any size of number. So, one set of students works on Task 1 (183 – 99 = 184 – 100) while another set of students works on Task 2 (583 – 199 = 584 – 200). In this case, the teacher provides one task suitable for students ready to work with three-digit numbers and a parallel task for students ready to work with smaller numbers. The strategic choice of the tasks still permits a meaningful class discussion that includes mathematical thinking generated from the parallel tasks.

Grade 3 Parallel Tasks

Task 1:		Task 2:
There were 583 students in Amy's school in the morning. 99 of the grade 3 students left for a field trip. How many students are left in the school?	⟷	There are 61 grade 3 students in Amy's school. 19 of them are in the library. How many grade 3 students are left in their classrooms?

By effective questioning and prompting, the teacher can help the students in both groups attend to the equivalence of the two subtraction situations. Some sample questions are as follows:

- How did you know that most of the students were left?
- How did you decide how many were left?
- I notice that Ian solved it by subtracting. Why does subtraction make sense?
- I notice that Lisa solved it by adding. Why might adding make sense?
- How would your answer have changed if one more student had left?
- How would your answer have changed if there had been one extra student to start with?
- How would your answer have changed if there was an extra student to start with, but one extra student left?
- Which problem is easier for you to solve?

It is through this kind of questioning and through the sharing of different students' approaches that students gain the guidance they need to respond independently to tasks that were previously too difficult for them to work alone.

Grade 6 Parallel Tasks

Task 1:		Task 2:
There were 10 625 athletes in the 2004 summer Olympics. Of these, 4329 were female. Calculate the number of male athletes. Estimate to check your work.	⟷	850 athletes participated in the Thames Valley Track and Field Event for Special Olympic Athletes. Of these athletes, 512 were female. Calculate the number of male athletes. Estimate to check your work.

The ultimate goal of differentiation is to meet the needs of the all students in a classroom during all parts of the problem-solving lesson. This becomes more manageable if the teacher can create a single task that allows not only different students to approach it using different processes or strategies, but also different students at different stages of mathematical development to benefit and grow mathematically. In this way, each student becomes a contributing and valued member of the classroom learning community.

How do I start down this road?

- Identify the key concepts for teaching and think through how they relate to overall and specific curriculum expectations.
- Find out where each of your students is on a developmental continuum or an instructional trajectory/landscape, related to those key concepts. Use different assessment strategies (e.g., observation, interview, performance assessment, work sample analysis) to gather evidence of your students' mathematical knowledge, thinking and experiences.

For Additional Strategies

Combined Grades: Strategies to Reach a Range of Learners in Kindergarten to Grade 6 (Ministry of Education, 2007)

"Differentiating Mathematics Instruction" LNS webcast at www.curriculum.org. (Select Webcasts for Educators under Popular Services on the left-hand side of the CSC homepage.)

Learn More about LNS Resources ...

Visit The Literacy and Numeracy Secretariat Guide to Print and Multi-media Resources at: http://www.edu.gov.on.ca/eng/literacynumeracy/PrintMultiMediaResources.pdf

Call:
416-325-2929
1-800-387-5514

Email:
LNS@ontario.ca

- Set learning and consolidation tasks in students' zone of proximal development. Use either open learning tasks (i.e., open-ended or open-routed) to provoke students to demonstrate their mathematics knowledge, skills and strategies. Use parallel consolidation tasks designed to suit the mathematical learning needs of different groups of students.
- Add elements of choice to your instructional plan, making sure that you use your instructional time wisely, enabling student learning to focus on important mathematical concepts and processes within students' zones of proximal development.
- Avoid the trap of believing that the teacher's job is to make each situation so clear and unambiguous that all students respond in the same way. A teacher who seeks to differentiate, in fact, wants sufficient ambiguity to enable lots of appropriate difference in responding.

What Matters Most

Many of the techniques described for differentiating mathematics instruction are useful. However, none are likely to be effective for student learning unless teachers consider the essential purpose of instruction, gather data using assessment for learning strategies and understand students' mathematical needs and readiness, individually and collectively.

References and Related Reading

Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership*, 7(3), 9 –22.

Dacey, L., & Lynch, J. B. (2007). *Math for all: Differentiating instruction, Grades 3–5*. Sausalito, CA: Math Solutions Publications.

Dacey, L., & Salemi, R.E. (2007). *Math for all: Differentiating instruction, K–2*. Sausalito, CA: Math Solutions Publications.

Fosnot, C.T., & Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.

Gregory, G.H. (2003). *Differentiated instructional strategies in practice*. Thousand Oaks, CA: Corwin Press, Inc.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Ontario Ministry of Education. (2005a). *Education for all: The report of the expert panel on literacy and numeracy instruction for students with special education needs, kindergarten to Grade 6*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education (2005b). *The Ontario Curriculum, Grades 1–8: Mathematics (Revised)*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education (2007). *Combined Grades: Strategies to reach a range of learners in Kindergarten to Grade 6*. Toronto: Queen's Printer for Ontario.

Ritchhart, R.(1999) Generative Topics: Building a Curriculum Around Big Ideas. *Teaching Children Mathematics*, 5, 462–468.

Schifter, D.,Russell, S.J.,& Bastable, V. (1999) Teaching to the Big Ideas. In Solomon, M. Z. (Ed). *The diagnostic teacher: Constructing new approaches to professional development*. New York: Teachers College Press, 22–47.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145.

Small, M. (2005). *PRIME: Professional Resources and Instruction for Mathematics Educators: Number and Operations*. Toronto: Thomson Nelson.

Small, M. (in press a). *Big ideas from Dr. Small*. Toronto: Nelson Education.

Small, M.(in press b). *Differentiating mathematics instruction K–8*. New York: Teacher's College Press.

Sternberg, R.J. (1985). *Beyond IQ: A triarchic theory of human intelligence*. New York: Cambridge University Press.

Tomlinson, C.A.(1999). *The differentiated classroom: Responding to the needs of all learners*. Alexandria, VA: Association for Supervision and Curriculum Development.

Tomlinson, C.A.(2001). *How to differentiate instruction in a mixed ability classroom*, 2nd ed. Alexandria, VA: Association for Supervision and Curriculum Development.

Tomlinson, C.A., & McTighe, J. (2006). *Integrating differentiated instruction and understanding by design*. Alexandria, VA: Association for Supervision and Curriculum Development.

Tucker, B., Singleton, A., & Weaver, T. (2002). *Teaching mathematics to all children*. OH: Pearson Merrill Prentice Hall.

Van den Heuvel-Panhuizen, M. (2002). *Realistic mathematics education as work in progress*. Retrieved July 5,2008 from http://www.fi.uu.nl/publicaties/literatuur/4966.pdf

Vygotsy, L.S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Western Australian Minister of Education and Training (2006). First steps in mathematics: Whole Number – Whole and Decimal Numbers and Fractions. Toronto: Pearson Education.