Ontario

Teaching and Learning Mathematics

The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario

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December 2004

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Une publication équivalente est disponible en français sous le titre suivant : *Enseigner et apprendre les mathématiques : Rapport de la table ronde des experts en mathématiques de la 4e à la 6e année.*

This publication is available on the Ministry of Education's website, at http://www.edu.gov.on.ca.

About This Report

The Expert Panel on Mathematics in Grades 4 to 6 in Ontario was established by the Ministry of Education to examine, synthesize, and outline the research to date on teaching mathematics to students in the junior grades (Grades 4–6). Panel members included teachers, consultants, principals, researchers, and professors.

There is now a growing body of research on mathematics instruction and learning in the elementary classroom, and this research has much to offer instructional practice in Ontario. The Expert Panel members read widely and discussed the research in detail, drawing out what they considered to be the essential ideas. They thought about these ideas in light of their shared knowledge of mathematics education in Ontario. This report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario is an examination of these essential ideas; it is a strong vision of effective mathematics instruction and support for students at the junior level. The report is for the benefit of all educators, all communities, and all children in Ontario.

Context for the Report

Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario continues an ambitious province-wide process of consultation and professional learning on mathematics education. It builds on the foundation laid down by other expert panels. Figure 1 shows the whole series of reports in context.

Figure 1: Recent Expert Panel Reports on Mathematics in Ontario Schools

Report	Focus
*Early Math Strategy: The Report of the Expert Panel on Early Math in Ontario (*Kindergarten to Grade 3*)* *Stratégie de mathématiques au primaire : Rapport de la table ronde des experts en mathématiques* (February 2003)	Developed by a panel of mathematics education experts (English and French language) to draw practical conclusions from current research about effective instruction in mathematics in the early school years in order to provide consistent, strategic guidance for educators. (*continued*)

Figure 1 *(continued)*

Report	Focus
Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario *Enseigner et apprendre les mathématiques : Rapport de la table ronde des experts en mathématiques de la 4e à la 6e année* (December 2004)	Developed by a panel of mathematics education experts (English and French language) to promote the mathematical literacy of students in Grades 4 to 6. This report builds on the foundations laid down in the report of the Expert Panel on Early Math in Ontario (2003) and complements the report of the Expert Panel on Student Success in Ontario (2004).
Leading Math Success: Mathematical Literacy, Grades 7–12 – The Report of the Expert Panel on Student Success in Ontario (May 2004) *La Numératie en tête de la 7e à la 12e année : Rapport du Groupe d'experts pour la réussite des élèves* (June 2004)	Developed by an English-language panel of experts to provide direction to Ontario school boards on mathematical literacy for at-risk students in Grades 7 to 12. Developed by a French-language panel of experts to provide professionals in French-language education with tools required to meet the mathematical needs of their students at risk in Grades 7 to 12.

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1 Introduction

> "It is the questions that drive mathematics. Solving problems and making up new ones is the essence of mathematical life."
>
> *(Hersh, 1997, p. 18)*

Mathematics is a fundamental human activity – a way of making sense of the world. Children possess a natural curiosity and interest in mathematics, and come to school with an understanding of mathematical concepts and problem-solving strategies that they have discovered through explorations of the world around them (Ginsburg, 2002). Yet, for many adults, the "sense making" of mathematics is lost. As educators we need to provide experiences that continue to foster students' understanding and appreciation of mathematics. By providing mathematics programs in which students explore and make sense of mathematical patterns and relationships, we can help students develop mathematical knowledge that allows them to solve problems and explore new ideas, in and out of the classroom.

Mathematics in the Junior Grades

The junior years are an important time of transition and growth in students' mathematical thinking. In Grades 4–6, the mathematics curriculum is changing in its content, sophistication, abstraction, and expectations of student proficiency. In terms of content, for example, the focus begins to shift from arithmetic to algebra, and data management broadens to include more probability.

In addition to this shifting of content emphasis, there is also a move to more abstract reasoning in the junior grades. Junior students investigate increasingly complex ideas, building on their capacity to deal with more formal concepts. For example, students learn to generalize patterns without having to draw each stage and record each term.

Junior students begin to make connections between different concepts and thus deepen their mathematical understanding. They learn to develop methods that can be applied in new situations. They begin to solve problems in more than one way. Junior students develop their ability to communicate their thinking to others orally and on paper. Finally, they are becoming more accurate in their work, both in reading problems and in recording solutions.

As students work with new ideas and concepts, they need programs and instructional approaches that will help them grow in their ability to use mathematics to make sense of their world – that will help them become more "mathematically literate".

Characteristics of Mathematically Literate Students

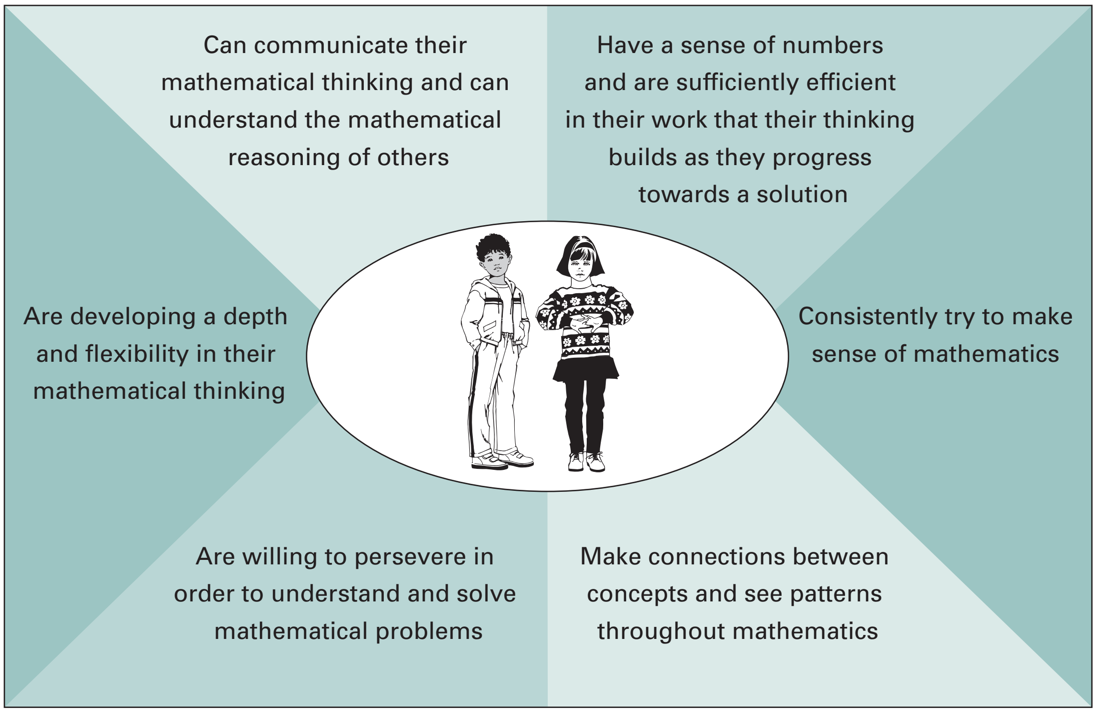
Mathematically literate students can be recognized by the following characteristics. They:

- ***consistently try to make sense of mathematics.*** Making sense is at the heart of mathematical literacy. Mathematically literate students understand what they are doing and why they are doing it. They have confidence that mathematics will assist them in solving everyday problems.
- ***are developing a depth and flexibility in their mathematical thinking.*** Mathematically literate students look at the numbers in a problem and think flexibly about how best to solve the problem. For example, when they see $1002 - 998 = ?$ they are likely to choose a method that is efficient and that is based on their *feel* for the numbers – perhaps adding up by 2 to 1000 plus 2 more to reach 1002 and find the difference of 4 – rather than working through the traditional subtraction algorithm using "borrowing". They are *thinking* about the numbers and how to best work with them.
- ***make connections between concepts and see patterns throughout mathematics.*** When these students learn a concept in one area, they can make use of it in new, related areas (Hiebert et al., 1997).
- ***have a sense of numbers and are sufficiently efficient in their work that their thinking builds as they progress towards a solution.*** As these students solve a problem, they work through intermediate steps that give them information they can use to solve the final problem (Russell, 2000). For example, if they are given the following problem: *What are all the possible dimensions of rectangular floor plans that could be made with 48 whole square tiles?*, they will begin with two numbers that are 48 or smaller, and after some exploration they will notice, in trying various pairs, that as the width of the floor increases, the length decreases. They will probably work in an organized rather than a random way, beginning perhaps with 1×48, 2×24, and so on. Their mathematics inquiry is not random, it is reflective; it builds and is therefore more efficient.
- ***are willing to persevere in order to understand and solve mathematical problems.*** These students enjoy mathematics and take pleasure in the insights they have as a result of their efforts (Gadanidis, 2004). They experience mathematics as a creative and interesting, though challenging, endeavour.

- ***can communicate their mathematical thinking and can understand the mathematical reasoning of others.*** "The heart of mathematics is the process of seeing relationships and trying to prove these relationships mathematically in order to communicate them to others" (Fosnot & Dolk, 2001a, p. 8).

These characteristics are summarized in the graphic below.

Characteristics of Mathematically Literate Students



For mathematically literate students, the junior years are a time of growing mathematical confidence, interest, and sophistication in the subject. For other students, however, the junior years can be a time of growing confusion – a time when they abandon their natural ability to think mathematically and to make sense of mathematical situations (Ginsburg, 2002). For them mathematics is neither sensible nor creative but rather a set of rules to be followed (Hiebert, 1999).

It is known that, initially, most children come to school as enthusiastic, curious thinkers, whose natural inclination is to try to make mathematical sense of the world around them. This natural curiosity can be nurtured in a problem-solving approach that begins with, and fosters, students' own ideas and methods. For example, Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) found that two-thirds of Kindergarten and Grade 1 students in mathematics programs focused on problem solving were able to solve the following problem: *If a class of 19 children is going to the zoo and each car can take 5 children, how many cars are needed?* When asked whether all the cars were full, they said: "No, there is an extra seat in one car" or "Yes, because I'm going too!" They were making sense of the question. Contrast these findings with test results of Grade 8 students in non-problem-solving programs[1] who were asked the same type of question, but with larger numbers: *An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?* Two-thirds of the 45 000 students tested performed the long division correctly. However, some wrote that "31, remainder 12" buses were needed, or just 31 – lopping off the remainder. Only one-quarter of the total group gave the correct answer of 32 buses (O'Brien, 1999). For those students, learning "school mathematics" (Fosnot & Dolk, 2001b) meant carrying out procedures without making sense of what they were doing. Is there evidence that Ontario students stop making sense of what they are doing in mathematics as they progress through school?

Mathematics in Ontario

Our overall reading of large-scale testing data combined with our knowledge of the field leads us to the conclusion that the province has made inroads into teaching mathematics in a more effective way, with a stronger emphasis on problem solving; we also find evidence of improved student understanding and achievement. For example, results from the 1999 Third International Mathematics and Science Study – Repeat (TIMSS-R), in which approximately 4000 Ontario Grade 8 students participated, are now available (Education Quality and Accountability Office [EQAO], 2000). The 1999 test included a majority of multiple-choice items, with some short-answer items and a few extended-response items across all strands of mathematics. Results of data analysis indicate that boys and girls fared equally well. Ontario students (both English and French) performed significantly better than the international average (based on the results of 38 countries). This was a statistically significant improvement from the 1995 results, which showed Ontario students performing at or below the international average, depending upon the strand (International Study Center, 1995).

On the other hand, the results of provincial assessment for some students at the Grade 6 level are not quite as encouraging. In 2003, 36% of Grade 6 students achieved at level 1 or 2 in the provincial assessment. It should also be noted that there was a negative shift in students' attitude towards mathematics. In the Grade 3 assessment three years earlier, 68% of boys and 60% of girls indicated that they liked mathematics (EQAO, 2000). By Grade 6, 55% of boys and 40% of girls reported liking mathematics.

Whether tests are international or provincial, they provide only a very limited view of students' mathematical literacy, which must be fleshed out with information from the classroom. Not surprisingly, teachers see a range of mathematical literacy among their

1. The results are from the National Assessment of Education Progress (NAEP), which is an extensive testing sample of students in the United States at regular intervals throughout their schooling. There is widespread consensus that most instruction at this time (the 1980s) would have been traditional direct instruction of mathematics as rules and procedures rather than instruction based on a problem-solving approach (Battista, 1999).

students: from a limited understanding of the mathematics, through to an ability to perform basic mathematics in a procedural fashion, through to a deep level of mathematical literacy. While Ontario has seen an improvement in mathematical understanding among many students, not all students have reaped such benefits. For some, mathematics remains a subject that they learn to fear and dislike as they move through the grades. Rather than developing mathematical literacy and confidence in their ability to do math, some students become less confident mathematically, learn to stop thinking mathematically, and come to rely on memorizing procedures to get correct answers. So although many positive changes have been made, there is more to be done.

How can educators continue to strengthen students' mathematical literacy? What can educators learn from their reading of the research, and how can those involved respond as a system (teachers, principals, parents, boards, and the ministry) to ensure that they continue to support the mathematical needs of Ontario students in the twenty-first century?

A heightened awareness of and shift towards teaching through problem solving is occurring in many Ontario classrooms. A factor contributing to this progression has been the quality professional development opportunities that have been provided for lead teachers of mathematics in the primary grades by the Ministry of Education. Anticipation of similar professional development experiences for teachers in the junior grades is growing. This report should provide the basis for the content of that professional development.

In the sections that follow, we will present our conclusions about instructional practices that are effective in enabling junior learners to develop a deep level of mathematical literacy. We consider various resources that may be used in a junior math program, and we look at factors that have a significant impact on the level of mathematical literacy that the junior learner can achieve, including basic attitudes and background experience. We examine the ways in which assessment supports students in their understanding of mathematics. Finally, we look at the crucial role of different professionals and organizations in Ontario in the continued development and support of strong mathematical literacy in our students.

We begin with an examination of effective mathematics instruction.

2 Effective Mathematics Instruction

"Clearly the twenty-first century requires a greater focus on a wider range of problem-solving experiences and a reduced focus on learning and practicing by rote…The decision requires in part a value judgment as to which needs are most important. But new research can also inform our choices."

(Fuson, 2003, p. 301)

A Balanced Program

An effective mathematics program should include a variety of problem-solving experiences and a balanced array of pedagogical approaches. An essential aspect of an effective mathematics program is balance (Kilpatrick, Swafford, & Findell, 2001). Students in the junior grades benefit from a varied approach that builds on their experiences in the primary grades and encompasses a balance of the following elements:

- ***Conceptual and procedural understanding.*** Junior students need instruction that helps them to develop conceptual understanding and also provides them with opportunities to practise and consolidate procedures that are meaningful and efficient for them.
- ***Skill development and problem solving.*** A balanced program provides rich problem-solving contexts that allow students to develop their understanding of mathematics and that give them opportunities to practise and consolidate skills.
- ***Lesson types.*** There should be a balance in the overall program that incorporates a variety of lesson types, such as the problem-based lesson described later in this section, as well as minilessons, games, and mental math.
- ***Instructional approaches.*** Three approaches to mathematics instruction (guided, shared, and independent) are interwoven throughout a balanced mathematics program. A detailed discussion of these approaches may be found in *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3* (2004). Each of these instructional approaches is featured in the classroom example beginning on p. 10.
- ***Groupings.*** A variety of groupings of students are planned for and integrated into an effective mathematics program to provide students with time to share ideas with their peers and to work independently.

- ***Assessment strategies.*** A variety of assessment strategies should be used so that all students have the opportunity to demonstrate what they know and what they can do in the ways that suit them best.

Effective Instruction: A Classroom Example

There is now an extensive body of research that finds that effective instruction, the kind that will develop strong mathematical literacy, also includes the specific characteristics listed below.

Characteristics of Effective Mathematics Instruction

Effective mathematics instruction:

- is focused on having students make sense of mathematics;
- is based on problem solving and investigation of important mathematical concepts;
- begins with the student's understanding and knowledge of the topic;
- includes students as active rather than passive participants in their learning;
- has students communicate and investigate their thinking through ongoing discussion;
- includes all students, whether in the choice of problems or in the communicating of mathematical ideas;
- incorporates ongoing assessment of student understanding to guide future instruction.

(Based on *Hiebert et al., 1997*)

Effective mathematics instruction must include a variety and a balance of pedagogical approaches. The type of instruction outlined in the list above is missing in traditional mathematics instruction, in which the teacher poses a problem, explains the process of solution in small, atomized steps, and then has students practise more of the same. Such instruction has proved insufficient for generating a deep understanding of mathematics for all (Battista, 1999). Many students may develop procedural fluency, but they often lack the deep conceptual understanding necessary to solve new problems or make connections between mathematical ideas.

Students cannot simply receive knowledge from the teacher and understand it in the way that the teacher thinks about it. As Lambdin (2003, p. 11) states:

> "A teacher's goal is to help students understand mathematics; yet understanding is something that one cannot teach directly. No matter how kindly, clearly, patiently, or slowly teachers explain, they cannot *make* students understand."

Instead, students must construct or "reinvent" mathematical ideas in order to understand them. In the view of Vygotsky (1986), concepts cannot be assimilated by the child in a ready-made form but have to undergo a certain development. Some teachers have interpreted Vygotsky's research as an injunction "not to tell" students how to do mathematics but instead to let them "discover" mathematics (Chazan & Ball, 1999). Discovery instruction has also, however, proved generally insufficient for many children (Askew, 1999). Even in rich mathematical environments children will not necessarily "discover" important mathematical concepts and understanding on their own. Moreover, if teachers are entirely hands-off, students may miss out on some important mathematics. Therefore, we are suggesting the use of instructional methods that will have some aspects of both of these approaches – procedures based and discovery – but will be fundamentally different from them and will encompass much more than either of them.

We are looking at instruction that uses a variety of instructional methods, which, taken as a whole, have strong evidence of improving students' understanding of and attitude towards mathematics.[2] We will illustrate these methods by way of an extended example of a whole-class lesson, which follows, and, later, by some additional lesson suggestions. As we do so, we will describe the research and reasoning that have led us to conclude that instruction based on a problem-solving or investigation approach is the means by which Ontario students will most readily achieve strong mathematical literacy.

The Specifics of the Lesson

Figure 2 on the next page is an outline of the specifics of effective mathematics instruction in the course of a whole-class lesson.

What follows is a discussion of each of these features:

Beginning of the lesson

- ***The teacher chooses a problem that offers a range of entry points for students at different levels.*** Every classroom contains students with a range of understandings and prior knowledge. Students who are given problems that are too difficult or too easy for them are not given the opportunity to learn from the experience. To avoid this difficulty, teachers need to choose problems that can be accessed in some way by the students.

For an example, we will turn to a lesson conducted in an Ontario classroom, on which we will draw throughout this report to illustrate what is meant by teaching through problem solving.[3] We have chosen a division lesson to illustrate the points in this discussion, but it is important to note that the methods discussed apply to **all strands** of mathematics, not just to number sense.

2. We anticipate that the details of such instructional methods will be outlined in the forthcoming *Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6.*

3. This lesson is drawn from a videotaped record of an Ontario classroom. The names have been changed and the lesson has been adjusted to create a model example of a strong learning context. The problem as it was originally posed can be found in Wickett and Burns, 2003, pp. 151–168.

Figure 2: Specifics of Effective Instruction in a Whole-Class Lesson

Beginning	• The teacher chooses a problem that offers a range of entry points for students at different levels. • The teacher poses the problem or sets the investigation without giving the steps for solution.
Middle	• Students work in pairs or in small groups to solve the problem. • Students work to make sense of the problem in their own way. They look for patterns and for connections with other problems. • The teacher asks careful questions that will help students to deepen and clarify their thinking. • Students communicate their mathematical thinking to one another, explain their ideas, listen to their peers, and talk with the teacher. • Students learn to persevere. • The teacher takes the necessary time, focusing on key ideas in some depth, rather than on a broad coverage of concepts. At least one hour a day should be allotted to mathematics instruction at the junior level. • Students and the teacher examine errors together as important opportunities for learning.
End	• Students share, explain, and examine a range of solutions with the whole class, discussing the common elements, looking for patterns, and making sense. • The teacher facilitates the sharing of ideas and discoveries in a community of learners. • The teacher organizes the discussion by choosing particular samples of students' strategies to build understanding of specific mathematical concepts and to support students' movement towards efficient methods.

The lesson takes place in a Grade 5 classroom towards the end of the school year. Ms. H teaches a class with a wide range of abilities, both in mathematics and literacy. In the first phase of the lesson Ms. H poses an opening problem for a unit on division: "I have a problem. I have a jar which I know held 317 marbles when it was full. As you can see, it is empty, and I want to fill it full of marbles again. At my corner store I can buy small bags of marbles, with 23 marbles in a bag." She holds up the jar and the small bag of marbles. "I want to go to the store and buy enough bags but no extras. How many bags should I buy?"

This problem can be solved through a wide range of methods and at various levels of mathematical sophistication. On the one hand, it can be solved by counting out concrete materials to add up groups of 23 to reach 317. It can also be solved in more demanding ways, through division without concrete representation or using many other intermediate steps.

Some students may access the problem at its simplest level and just count each marble. As they move through the process, through discussion with other students and during the "reflect and connect" part of the lesson, they will begin to recognize that there may be more efficient and effective ways of tackling the problem. Other students, who may have a stronger understanding of the concept, can approach the problem more efficiently from the beginning by looking for patterns or using other more strategic approaches to the problem. To be able to choose problems with multiple entry points means that the teacher needs to hypothesize about the students' present level of knowledge and connect that to the learning goals. Open-ended questions – in this case, questions with multiple solution routes – offer the best framework for accessible problems for all students.

• ***The teacher poses the problem or sets the investigation without giving the steps for solution.*** Over the last two decades, teachers have increasingly taken on the challenge of incorporating problem solving into most aspects of their instruction. They have been shifting their practice to include problems that are more than simply exercises at the end of a lesson or a unit. They are also helping students to learn strategies for solving various types of problems. While the use of problem solving as the vehicle for instruction includes these practices, it can involve much more: it can be used as the means of *introducing concepts* rather than of simply engaging students in applying or practising mathematical procedures or following steps taught at the start of a lesson. "Problems to be solved can be used effectively as a context for students to learn new concepts and skills, not just as applications of previously learned skills" (Kilpatrick, 2003, p. 17). Students learn concepts and skills more deeply through a problem-solving approach, when the ordinary steps to solving the problem are not taught at the beginning of the lesson.

Middle of the lesson

• ***Students work in pairs or in small groups to solve the problem.*** Paired or small-group work gives students a time to explore their own thinking without having input from the rest of the class. It gives them a time to try out a variety of ideas with only one or two other students. Students need this time to pursue their ideas and to begin to clarify and to verify their thinking through discussion. In addition, the paired or small-group work can be the source of another point of view for students when they hit the inevitable roadblocks.

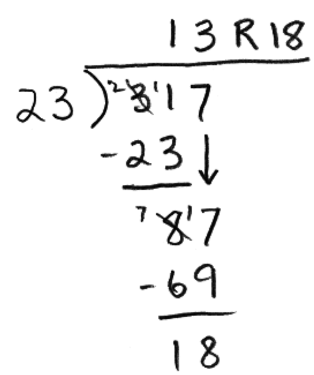
Ms. H has not yet taught a standard algorithm for division, although a few students have seen it at home. Rather than going through the steps of division, students talk about the problem and look at the jar and the small bag of marbles. Then they return to their seats to work in pairs. Each pair has a jar and a small bag of 23 marbles.

• ***Students work to make sense of the problem in their own way. They look for patterns and for connections with other problems.*** It would seem most efficient to teach students a standard set of procedures (in this case, the traditional division algorithm) before solving the problem. Yet for many students such instruction results in a superficial knowledge of mathematics, rather than a deep knowledge of why the rules work and of how to use them. Often the efficient rules in mathematics – for example, the steps for division or the rules for finding the volume of an object, or even the mathematics underpinning mathematical tools such as a metric ruler – are initially too abstract for most students to learn through transmission.

In the case of algorithms, the level of abstraction is particularly challenging for students. Historically the algorithms (standardized steps for calculation) were created to be used for efficiency by a small group of "human calculators" when calculators were not yet invented (Ma, 2004). They were not designed to support the sense making that is now expected from students. For example, in order to make use of the standard algorithm for division in use in North America, students must treat each digit separately. In solving 317 ÷ 23=? they would normally think, How many times does 23 go into 3? (rather than 300), then, How many times does 23 go into 31? (rather than 310) (see Figure 3).

Although the standard algorithm is very efficient, when it is taught before students understand the concept of division or have a solid understanding of place value, students are forced to abandon their sense making of the question, and their results may be senseless mathematics, as in students' common responses to the army bus problem previously described. When, however, instruction begins with what students know, and the teacher works with students' ideas and methods before introducing formal rules, students understand the concepts more deeply. Moreover, most students taught in this way make fewer errors and their errors are more "sensible" and easily corrected than is often the case when children try to follow rote procedures (Carpenter, Fennema, Franke, Levi, & Empson, 1997). Most teachers of multidigit algorithms have experienced the "wild" answers that students give when they do not understand what they are doing. As one frustrated student tersely informed researchers inquiring about her wildly incorrect answer: "That's the way it *has* to be done. This is the way I learned it in school, so it

Figure 3: Standard Division Algorithm



has to be the way" (Baroody & Ginsburg, 1990, p. 63). Where instruction begins with student methods and works with student ideas to move towards more effective methods, students keep *thinking* about the mathematics, trying to make sense of their work, and eventually their understanding deepens. The mathematics does not end with the students' own methods, but it should begin with them.

> As students grapple with the problem, they typically begin by trying to make sense of the different components – that is, how to use the information (a bag of 23 marbles and the total number of marbles in a full jar) to find their answer. As the pairs begin to work, some students work by themselves for a while and then begin discussing their thinking with their partner. Other pairs immediately begin a dialogue. The class is noisy, and there is a great deal of gesturing towards marbles and jars on the table, pouring of marbles into the jars, and writing of calculations and diagrams in mathematics books. The nature of the talk between the students in the pairs varies. Some students explain their ideas and convince their partner to use their method. Other pairs of students explain and ask questions but work through the solution in their own way, conferring back and forth.

• ***The teacher asks careful questions that will help students to deepen and clarify their thinking.*** This is not quizzing by the teacher to get answers that are already known (Chapin, O'Connor, & Anderson, 2003). Rather, it is an effort to find out what the students are thinking and how they are thinking about it. Questioning is a vehicle for supporting the development of students' ideas. Students' responses also provide the teacher with information that is helpful in developing next steps in instruction for individuals or the whole class.

> Ms. H circulates among the students, asking them to explain and probing with questions like the following: "Why do you think…?" "How do you know?" "What does this number mean?" "Can you explain your partner's reasoning?" "Do you agree or disagree with their reasoning and why?" " Why doesn't this solution work?" "Could you solve it in another way?"

• ***Students communicate their mathematical thinking to one another, explain their ideas, listen to their peers, and talk with the teacher.*** Communication of ideas plays an essential role in the development of strong mathematical understanding in students. Traditionally, communication of mathematics took the form of the teacher questioning and one student answering, the teacher questioning and another student answering, and so on (Forman, 2003). The students' answers may or may not have included an explanation. Today, there is extensive evidence that if students are engaged in mathematics communication in which they are expected to explain their ideas clearly and follow other students' reasoning (rather than just the teacher's instruction), they are much more likely to develop a deep understanding of the concept. In longitudinal research conducted in several schools in Ontario, for example, Radford (2000) found that environments featuring rich discussion of this type allowed students to engage in the re-creation of important underlying mathematical ideas. This type of communication of ideas is at the heart of strong mathematics classrooms.[4]

In Ms. H's class the students continue their work for some time. Many make errors, and they find on the whole that the problem is a real challenge. Some students make several false starts. Occasionally Ms. H will stop the class to discuss a point or perhaps reassure the students that she expects them to be thinking hard and that it's okay to sometimes be confused, since this is part of the process. Most of the mathematics communication, however, is done among the students as they work in pairs and sometimes in groups to solve the problem.

• ***Students learn to persevere.*** Genuine mathematics, in which students engage deeply with problem situations, can be a challenge. Solving a problem may take longer than one class; students may experience moments of frustration. Traditionally, teachers have tried to make this easier for students by laying out the path in small, incremental pieces. However, students will often learn more deeply if they experience moments of hard thinking, followed by the satisfaction of finding solutions to the problem. Further, "getting stuck" and learning how to persevere – and how to change tactics when necessary – is a fundamental aspect of learning how to work mathematically. Even among working mathematicians, learning how to deal with getting stuck is a very common experience. Leone Burton (2004) reported that 55 out of the 70 working mathematicians whom she interviewed discussed struggle as part of their work. As one mathematician reported: "The natural condition of doing [mathematics] research is to be stuck, most of the time, on most of the things you are doing" (p. 59).

• ***The teacher takes the necessary time, focusing on key ideas in some depth, rather than on a broad coverage of concepts.*** Working with challenging problems takes time. If students are to work with mathematics in a way that is likely to produce deep understanding, they need sufficient time to grapple with the math. The allotment of sufficient time is a fundamental component of programs that foster mathematical literacy. **At least one hour a day should be allotted to mathematics instruction at the junior level.**

• ***Students and the teacher examine errors together as important opportunities for learning.*** In effective instruction, errors are also treated differently from the norm – that is, rather than being viewed as problems in need of direct correction, errors are viewed as points of discussion and opportunities to learn. The teacher asks students to make sense of why certain methods do not work.

In addition to being allowed to make and discuss errors, students need to learn how to verify answers for themselves, rather than relying solely on the teacher as the final arbiter.

4. Stigler (2000), for example, in his examination of instruction in Japanese classrooms, found that there was significantly more, and more complex, explanation of thinking by both the teacher and the students than in American or German classrooms. He cited this as a key aspect of the success of Japanese teachers in engendering a strong understanding of mathematics in their students.

They need to learn how to judge whether their answers make sense and whether they are possible (Flores, 2002).

> As the range of abilities in Ms. H's class is large, the students eventually solve the problem using a wide variety of methods and displaying various levels of mathematical sophistication. Some of their first attempts to answer the problem may be erroneous or inefficient. However, as they work through the lesson (and future related lessons), they will examine and discuss their errors and make strides in increasing the efficiency of their methods.

End of the lesson

• ***Students share, explain, and examine a range of solutions with the whole class, discussing the common elements, looking for patterns, and making sense.*** A thoughtful discussion of the commonalities and differences between the solutions allows students to focus on the structures of mathematics. This discussion helps them to organize their thinking, clarify it for others, and modify it if necessary. It also allows them to see how other students have approached the same problem and how the different methods are mathematically linked, and thus broadens and deepens their understanding of the mathematics.

Students must learn to explain their ideas not only orally but also in written form. They learn to write up their ideas in an organized fashion, so that both the teacher and other students can understand their reasoning. They learn to use a variety of forms to explain their reasoning and to convince others of the soundness of their thinking. They learn to use mathematical terminology and symbolism in order to communicate effectively with the rest of the class and at the same time improve their own understanding. They learn to do these things in the service of trying to communicate their ideas to their peers and to the teacher.

> Students begin the third part or end of the lesson by putting up their solutions on chart paper or on the board. The class reassembles to discuss the various solutions. Ms. H has already looked at the solutions and has decided which ones she wants to have shared, on the basis of the discussion she thinks they will generate. The students share their answers, and students question one another about the various methods of solving the problem. It is apparent that the presentations are not just for the teacher; students question one another and ask for clarification. There is discussion among the students in which Ms. H. stands back. At other times Ms. H interjects with questions to draw out the important mathematical concepts and to support students in the detail and clarity of their explanations and their questions.

• ***The teacher facilitates the sharing of ideas and discoveries in a community of learners.*** The creation of a community of learners is essential to an effective mathematics program. In a community of learners, students feel comfortable discussing their errors, try to understand other approaches, and work together to better understand concepts. The community of learners must provide a risk-free environment – that is, an environment in which students are free to take risks, to share their ideas, solutions, and thoughts. A teacher establishes a sense of community, respect for others, and a risk-free environment by modelling a positive attitude towards mathematics, accepting "mistakes" as opportunities for learning, fostering positive critical thinking, creating displays of student math work, and acknowledging and building on the unique ways in which each student thinks.

The development of a community of learners, especially in the mathematics classroom, allows students and teachers to work together towards developing understanding. In a community of learners the teacher is no longer the sole source of expertise. With the establishment of a community, students are better able to engage in productive mathematical exploration and discussion.

The kind of communication encouraged in a community of learners is a skill that students need to learn. Teachers will need to spend time brainstorming with students about what good communication looks like (e.g., partners look at each other when they talk) and what it sounds like (e.g., "I'm not sure I understand what you mean. Can you explain it again?") Some teachers have students act out or model a good math discussion – including a worthwhile disagreement – in order to support students' ability to work as an effective mathematics community.

- ***The teacher organizes the discussion by choosing particular samples of students' strategies to build understanding of specific mathematical concepts and to support students' movement towards efficient methods.*** During the end-sharing part of the lesson, students will describe and discuss their thinking and strategies. Unanticipated mathematical ideas may arise that the teacher considers worth pursuing in depth. Beyond this, however, the teacher hopes to draw out of the discussion the concept that was the reason for choosing the lesson.

> A few pairs of students try to determine the number of bags needed by measuring the height of one bag of marbles in the jar and then multiplying up or dividing to find how many bags are needed to reach the top of the jar. This is an unanticipated method that interests the students, and Ms. H. decides to pursue discussion of it. While the mathematics these student pairs use is accurate, their final answer is not as accurate as the answer found by those groups who begin with the total number of marbles needed. The class discusses why the method of measuring was less accurate than other methods using the total number of marbles.
>
> As students continue to share, Ms. H also draws out of the discussion one of the *big ideas* of multiplication and division – that is, that they have an inverse relationship and therefore one can both multiply (or add up) to find the solution or, more elegantly, divide (or repeatedly subtract) to find the answer. For example, pairs of students have written their solutions on the board, and Ms. H asks questions about two of the solutions (see the examples below) to focus the discussion: "How is it that Maria and Paula multiplied [example B] and Parmvir and Raj divided [example C] and still got the same answer?"

Some Solutions to the Marble Problem

Example A: Doubling Strategy

We have to find out how many bags of marbles with 23 marbles in each bag will fill a jar with a maximum capacity of 317 marbles.

23 > 46
23 > 92
23 > 46
23 > 184
23 > 46
23 > 92
23 > 46
23 > 276 > 299 > 322
23 23
23 > 46
23 > 92 > 92
23 > 46
23

The number of bags you would have buy is 14. You would have to then use 317 of the 322 marbles, and be left with 5 marbles.

Example B: Multiplying-Up Strategy

We have to find out how many bags of marblos are in this jar. We know that each bag has 23 marbles as the jar can hold a maximum of 317 marbles.

$$\begin{array}{r} 23 \times \underline{10} = 230 \\ \underline{2} \times 23 = \underline{\ \ 46} \\ 276 \\ \\ \underline{1} \times 23 = \underline{\ \ 23} \\ 299 \end{array}$$

← not enough to fill capacitiy

$$\begin{array}{r} \overset{2}{\cancel{3}}\overset{0}{\cancel{1}}{}^{1}7 \\ -\underline{299} \\ 18 \end{array}$$ marbles

You would need 14 bags to fill the jar.

Example C: Continental Division Algorithm

We have to find out how many bags of marbles can fit in the jar.

We know that the jar can hold 317 marbles and every bag holds 23 marbles.

```
      13 r. 18.
23) 317
  - 230   | 10
     87
   - 46   | 2
     41
   - 23   | 1
     18
```

If you only have 13 bags (299 marbles) that's not enough to fill the jar. So you need another bag to fill the jar. But you have 5 left over.

Planning and Knowledge

The description given in the preceding pages has delineated some of the detail of teaching a whole-class lesson using a problem-solving or an investigation approach. One can appreciate that such a lesson does not simply "happen" but is the result of planning as well as of a strong knowledge of both mathematics and pedagogy on the part of the teacher. The following is an outline of some aspects of the planning and knowledge needed to undertake this kind of instruction (see Figure 4 on p. 19).

• ***The teacher chooses a lesson with the big ideas of the mathematics topic in mind.***
Many teachers are justifiably concerned about whether the mathematics lessons that they present focus on the important concepts that students need to know for future mathematical success. While it would seem common sense that most of the mathematics taught in various text and non-text lessons are built on mathematics concepts important for mathematical development, this unfortunately is not always the case. In the TIMSS video analysis of the mathematical content of observed lessons in the United States, Germany, and Japan there was a wide discrepancy in the quality of mathematics content.

Figure 4: Some Aspects of Planning and Knowledge Needed for Instruction

Planning	• The teacher chooses a lesson with the big ideas of the mathematics topic in mind. • The teacher is aware of the many strategies students might use to solve the problem in the lesson and of the foundational big ideas that underlie these strategies. • The teacher anticipates misconceptions that students might have about the mathematics in the lesson and prepares to address them. • The teacher chooses the next lesson to build on what students know and to direct them towards making new connections to extend their knowledge.

In more than half the Japanese lessons, the quality was judged as high; in almost all of the U.S. lessons, the content quality was low (Stigler & Hiebert, 1999). The judgements were based on whether important mathematical concepts were addressed in the lesson.

When teachers have curriculum that is structured and clustered around the essential topics in mathematics and, further, underpinned by the big ideas within each topic, they have one possible way to determine what constitutes lessons that are likely to lead to the learning of important mathematical ideas. For example, within the topic of multiplication and division there are a number of big ideas.[5] When students construct a big idea, it is big because they make connections that allow them to use mathematics more effectively and powerfully (Fosnot and Dolk, 2001b). The big ideas are also critical leaps for students who are developing mathematical concepts and abilities.

In the case of Ms. H's class, when students have constructed the following understanding – that:

the total number of marbles ÷ the bag of 23 marbles = the number of bags needed

is the inverse of

the bag of 23 marbles × the number of bags needed = the total number of marbles

– they have constructed the big idea that multiplication and division have an inverse relationship. They will be able to use this knowledge to solve a range of problems flexibly and will be prepared to work with the concept of equality in algebraic reasoning. Students who construct this understanding of the inverse relationship know that $? \times 23 = 317$ can

5. The term *big ideas* was coined by Schifter and Fosnot in the course of their professional development work with teachers who were learning mathematics more deeply. Schifter and Fosnot defined big ideas as the "central, organizing ideas of mathematics – principles that define mathematical order.... [T]hey are connected to the structures of mathematics" (1993, p. 35).

be solved by $317 \div 23 = ?$ Or later, using algebra, they will understand why, if $23x = 317$, they can then solve for $x = {}^{317}/_{23}$. When children construct these ideas and make these connections, they are working creatively and flexibly with numbers, which activity is at the core of what mathematicians do.

> In addition to focusing on this idea, Ms. H also intends to look at some of the different strategies students have used, in order to support student movement from some of the more inefficient methods (e.g., adding up by doubling 23 to reach a total of 317 to find the number of bags of marbles needed [example A, p. 17] towards the more efficient methods of multiplication [example B, p. 17] and, more efficient still, division using subtraction of groups of 10 [example C, p. 18]).

• ***The teacher is aware of the many strategies students might use to solve the problem in the lesson and of the foundational big ideas that underlie these strategies.*** If teachers instruct by starting with student thinking, they must have some knowledge of the progression of student strategies in a given mathematics topic (Baek, 1998). One of the goals in mathematics is to have students eventually become fluent and flexible in their mathematical calculations and in the application of rules while at the same time continuing to understand what they are doing. In order to foster student movement towards more efficient methods of mathematics and do so with understanding, teachers need to have information on the progression of strategies that students might use within a given topic. In addition, they need information on what these strategies indicate about students' understanding of the mathematics concepts.

There is a very loose progression of strategies that students use. In solving division problems, for example, these may range from initial strategies based on concrete modelling through to the efficient and flexible use of some type of division algorithm. The progression is not linear; it varies from student to student, with some students "skipping" typical benchmarks and others diverging along the way to investigate tangential ideas. It is also *highly dependent upon the instructional methods used*. This progression of strategies is unlikely to appear in classrooms unless instruction begins with student ideas.

• ***The teacher anticipates misconceptions that students might have about the mathematics in the lesson and prepares to address them.*** As teachers become familiar with students' various solution methods, they also develop a knowledge of the typical misconceptions that students will hold and can prepare to address them productively during the discussion.

> Ms. H raises the issue of the remainder by posing the question: "I notice that Parmvir and Raj [example C] divided for an answer of 13 R18 but said they needed 14 bags. Why did they change their answer to 14?" The students debate about the "math" answer of 13 R18 and the real-world context of the problem – the reality that no store is going to break open a bag and sell only 18 marbles from it. They discuss what the R18 means and agree that 14 bags are necessary. They would not say that 13 R18 bags should be purchased,

because they are engaged with the problem and trying to make sense. While they have sorted out the issue of the remainder in this problem, it is one they will address in many different contexts – Ms. H knows that children struggle with what the remainder means. She also wants to lay the groundwork necessary for understanding how and when to continue the division so that an answer with decimals results.

• The teacher chooses the next lesson to build on what students know and to direct them towards making new connections to extend their knowledge.

In a subsequent lesson Ms. H will introduce the structure of one possible algorithm (the Continental standard algorithm) using student information. One pair of students has already used this method (see example C on p. 18) as a result of their discussion with the teacher. (They were using an unstructured form of this method on their own but needed help to write it in an organized way.) Ms. H will take example C, discuss how the pair of students used the Continental algorithm, and discuss how other students (who are at this stage of mathematical development) can use it to organize their thinking in a manageable and efficient way.

Teachers who are reading and evaluating this example of a Grade 5 division lesson might consider its merit by asking whether it makes for a "good mathematics story" (Gadanidis & Hoogland, 2003). Does it resonate with students? Does it offer opportunities for students to gain mathematical insight?

Effective Instruction: Other Lesson Examples

An effective mathematics program encompasses a balance of instructional approaches within a variety of learning opportunities. The preceding lesson includes two of the three approaches of shared, guided, and independent mathematics. In the other types of learning opportunities that students should be able to experience in the mathematics classroom, one or other of these components predominates. For example, in minilessons such as the one described on the next page, the focus is on guided mathematics. In lessons in which students work cooperatively to solve puzzles or play math games, the emphasis is on shared mathematics. Some lessons focus largely on independent student work, and some lessons include opportunities for students to practise the mathematics concepts that they developed previously. A discussion of some of these other learning opportunities or types of lessons follows, beginning with a consideration of occasions for practice in rich contexts.

Rich Contexts for Practice

Students need opportunities to practise or "routinize" their mathematics learning in rich contexts. If students use their own methods to solve problems and teachers then scaffold these methods to more efficient but accessible methods, students are more likely to understand what they are doing. Most adults learned rules and procedures by memorizing first and then perhaps understanding later. In this document the approach taken is that effective instruction works with students' ideas first, supports the growth of students' understanding and development to the point of efficiency or rules, and then offers rich opportunities for routinizing their procedures and learning. These opportunities should be both to practise and to continue to explore mathematics. Practice in the absence of continued exploration by students is not an effective use of precious mathematics time. We agree with Fuson (2003), who suggests that "drilling for long periods on problems involving large numbers seems a goal more appropriate to the twentieth than the twenty-first century" (p. 302).

Mental Math

Mental math skills can be developed and practised in guided minilessons. Mental math involves calculations done in the mind, with little or no use of paper and pencil or calculator. It is an essential component of effective instruction at the junior level. Guided minilessons are opportunities for students to work mentally on various calculations or problems. Students may be writing some numbers down to keep track, but the goal is to have them work with numbers in a flexible way, using mental math. As students develop their ability to use mental math calculations, they often use the big ideas in mathematics. For example, in the case of multiplication of more than one digit, students work through problems to construct the big idea that numbers can be broken up into parts and multiplied and the products of the parts can then be added together to get the total product (the *distributive property*) – for example, $12 \times 13 = (6 \times 13) + (6 \times 13)$. They know that as long as all the 12 groups of 13 are accounted for, the sum of the products of the parts is the same. Teachers can support students' ability to play with the essential structures of mathematics in this way by working with related strings of calculations in minilessons. Looking at another classroom example (Fosnot & Dolk, 2001b, pp. 106–107) will clarify this point.

> Grady wants to strengthen his students' understanding of the distributive property and give the class a chance to practise its use. He writes down the strings of related calculations found on the next page. Most students know the mental answer to the first calculation. Some have memorized it; others think 10×6 and then divide 60 in half to get the answer. They also know that 30×6 is similar to $3 \times 6 = 18$ but 10 times as big. However, they struggle with 35×6. Most students double 35 to get 70 and then add that three times (35×6 is same as 70×3). However, someone else has a shorter way: that student simply adds the products of the first two calculations in the string, 30 and 180. The students discuss this strategy and try it out on the next string of numbers to see whether it works. This is not an extended lesson but a minilesson for practice. It is similar to a writing conference on a specific topic in that it is an opportunity to focus on one specific idea over a series of examples. This is often only a 15-minute lesson.

A Mental Math Minilesson to Develop the Distributive Property

Grady's Strings

5x6
30 x 6
35 x 6
2 x 7
40 x 7
42 x 7

Mental math is also sometimes used by students to communicate their thinking or to explain how they solved a problem. For example, some of the students in Ms. H's class explained to the rest of the class that they estimated how many bags of marbles would probably be in the jar by doubling mentally: "I doubled 23 to make 46, and 46 to make 92, and 92 to make 184, and then I knew the next one would be too big because 184 is close to 200." Students will use their mental math skills both to estimate and to check the reasonableness of their answers.

Math Games and Puzzles

Math games and puzzles provide a rich context for practising mathematics concepts. Well-designed math games are another way for teachers to have students hone their proficiency with numbers (Kamii, 1994). As Wickett and Burns state about their inclusion of games in the development of division concepts: "Our goal is to give additional alternatives for providing students much needed practice with division while also continuing to engage them in thinking and reasoning" (2003, p. xvii). It should be noted that these are games that support the development of specific mathematics concepts.

While games and puzzles can be an effective support for mathematical literacy in the Number Sense and Numeration strand, they are also useful in other strands such as Geometry and Spatial Sense or Data Management and Probability. The example that follows shows how a cooperative puzzle is used in the class to give students practice in using terms from geometry and measurement.

Ms. S is working on the development of students' understanding of definitions of various geometric and measurement terms in context by using the cooperative puzzles from *Get It Together* (Erickson, 1989, p. 53) called "Stick Figures 3" (see p. 24). Students are seated in groups of four, with each person in the foursome holding a clue to the one figure they must build as a group. (The remaining two clues are turned face down on the table to serve as extra clues if the students need help.) Ms. S knows that as students try to solve their problem, they will need to understand the terms used in the problem in order to find the answer. As students solve a series of cooperative puzzles, they are clarifying the meaning of mathematical terminology and getting practice in using it.

Cooperative Puzzle in Geometry

Stick Figures 3

There are twelve sticks in the figure. The sticks are unbroken and they don't overlap.

Make the figure!

Stick Figures 3

The figure is made up of two triangles that are not congruent.

Make the figure!

Stick Figures 3

One triangle has a perimeter of seven sticks.

Make the figure!

Stick Figures 3

No segment in the figure is shorter than two sticks.

Make the figure!

Stick Figures 3

The two triangles share a side.

Make the figure!

Stick Figures 3

There is a quadrilateral in the figure that has a perimeter of nine.

Make the figure!

Independent Work

Students should also have time to work independently. They might be working on:

- writing their own word problems, which they can later share with a math partner and then with the full class (see Silverman, Winograd, & Strohauer, 1992, for a description of setting up a class for student-generated problems);

- a more extended math project such as the following: *Keep track of the time you spend throughout the week on various activities such as eating, sleeping, TV watching, school, and so on. On average, what percentage of your time is spent each day on each activity? Create a pie chart to show how, on average, you spend your day. How does your use of time compare with that of another family member?;*
- "small interesting word problems" to strengthen their use of concepts already explored in more extended lessons (e.g., the TOPS problem-solving word decks A, B, C, and D published in 1980 by Dale Seymour Publications).

In this document, the discussion so far has been about the different types of lessons and instructional methods used in an effective program. Warranting separate discussion are various resources that are used throughout the junior math program: manipulatives, children's literature, texts, teacher's resource books, and technology.

The Effective Use of Resources Throughout All Classes

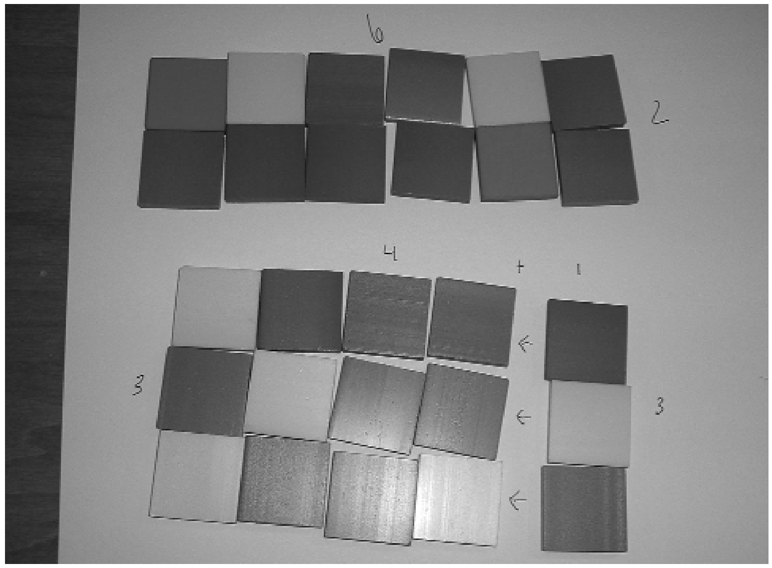
Throughout all of their math classes teachers will be making use of various resources. While we will look at each resource in turn, the overriding message in all areas is the same: effective instruction depends on both the quality of the resource and the skill of the teacher.

Manipulatives

Manipulatives that are used well are central to effective instruction and have the capacity to greatly improve and deepen student understanding. Manipulatives allow students and teachers to discuss something that is concrete; they also offer material upon which a student can act (Thompson, 2002). For example, students may struggle as they develop their understanding of the relationship between area and perimeter, and they may assume that as you increase the area of a figure, you necessarily increase its perimeter. Square flat tiles provide an effective concrete way to explore this idea. The teacher may pose a question: "I have a garden that is 6 m long by 2 m wide. I want to expand the area of my garden without buying extra fencing. Is that possible?" Students can use 12 square tiles to represent the garden and then experiment with adding tiles to answer the question. See Figure 5 on the next page.

In addition to providing a medium for experimentation and discussion, manipulatives can also provide a model or visual for complex concepts. Consider, for example, many adults' conception of the Grade 6 concept of the volume of a rectangular prism. When asked, "How many whole blocks measuring 2 cm by 2 cm by 2 cm could fit in a box measuring 5 cm by 6 cm by 4 cm?", many adults use the formula for volume to calculate $6 \times 4 \times 5 = 120$ cm^3 and then divide their answer by 8 cm^3 to get 15 blocks. These same adult students often have difficulty drawing the problem. Only when they build the box, tape centicubes into $2 \times 2 \times 2$ blocks, and place the blocks into the box do they

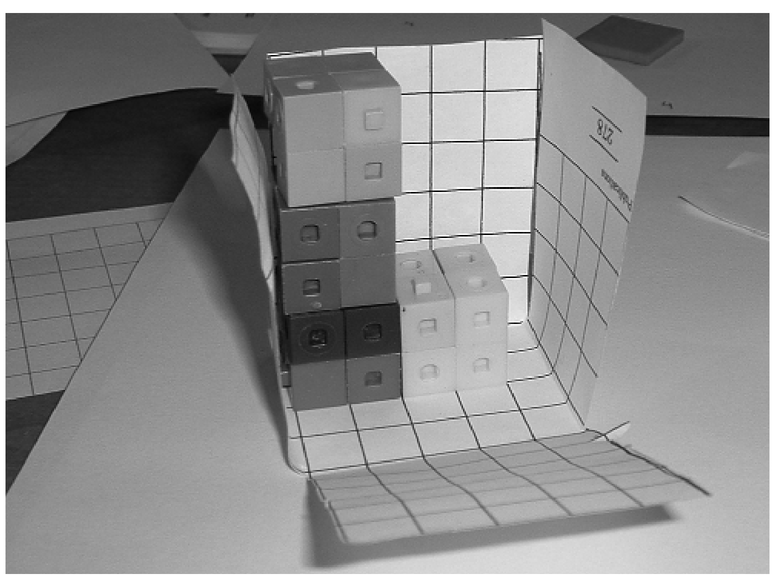
Figure 5: Experimentation With Manipulatives to Shed Light on a Challenging Concept



correct their answer. “Oh, I *see* it now!” (See Figure 6 below.) This concrete experience with manipulatives helps them to visualize the solution. Manipulatives provide the necessary bridge that earlier paper-and-pencil instruction did not. Similar results are found for junior and intermediate students (Battista, 2003).

Students learning in classrooms where manipulatives are used often outperform those in classrooms without manipulatives (Clements & McMillen, 2002). Manipulatives by themselves, however, do not generate understanding. In order to use manipulatives effectively the teacher must first select them with care, asking: “What, in principle, do I want my students to *understand*? [rather than] “What shall I have my students learn to *do*?” (Thompson, 2002, p. 246). The manipulatives chosen should be ones that allow students to use their own informal methods to solve a problem. Different students may

Figure 6: Building With Manipulatives to Visualize a Problem



choose different manipulatives, depending on how they think about the problem, and they should use their chosen manipulative to solve the problem and to explain their thinking. Manipulatives should be used by students to support their thinking, rather than by the teacher to demonstrate procedures.

Interestingly, some manipulatives and mathematical tools are more useful when they are initially constructed by the student. It is through the construction process that students build a deep understanding of the concept. For example, Marilyn Burns (2001) contends in her instruction on fraction concepts that of all the lessons she has created, it is the actual construction of the fraction kit that has proved most effective (the teacher asks, "How do we make a whole that is divided into two equal pieces? three equal pieces?" and so on). Researchers agree, proposing that students "reinvent" mathematical manipulatives and tools such as simple timing devices for understanding the measurement of time (Kamii & Long, 2003) or rulers for understanding standardized linear measurement (Young & O'Leary, 2002).

Manipulatives may be initially constructed by students, they may be "found" materials (e.g., boxes, paper bags, stir sticks, buttons), or they may be manufactured materials (e.g., square tiles, centicubes, geometric solids). A detailed list of recommended manipulatives can be found in Appendix A. **Manipulatives should be available in every junior classroom in sufficient numbers to support instruction.**

Children's Literature

Good mathematical problems emerge from a variety of contexts: mathematical contexts, physical contexts, real-life contexts, and imaginary contexts. Good mathematical problems draw students' mathematical attention and offer students opportunities to experience the pleasure of mathematical insight (Gadanidis, 2004). Literature can supply wonderful contexts for problems. It can spark the imagination of students – for example, reading *How Big Is a Foot?* (Myller, 1990) can engage students in the reinvention of standardized measurement. It can be used for humour – for example, reading *If You Hopped Like a Frog* (Schwartz, 1999) can lead to asking students, "If I were a chameleon, and a chameleon's tongue is half its body length, then how long would my tongue be?" (Wickett & Burns, 2003). Some teachers use centuries-old riddles both to promote interest in the subject and to develop mathematical thinking and debate – for example, by posing a riddle like the following: The storyteller and a wise mathematician were travelling on a single camel in the desert when they encountered three arguing brothers. The brothers were arguing because their father had left them 35 camels to be divided among them, with the oldest brother receiving $^1/_2$, the middle brother $^1/_3$, and the youngest brother $^1/_9$ of the camels. How might the mathematician help the brothers to solve the problem? (Bresser, 1995). Literature is a way to bring life and context to a problem – a way to help students see mathematics everywhere.

Textbooks, Teacher's Guides, and Other Professional Resources for Teachers

While the textbook is only one resource among many, it can nonetheless play an important role in an effective classroom. In-depth learning is more likely to happen when texts and curriculum are in alignment and concentrate on a few topics and the big ideas or important mathematics in these topics. Good textbooks that support effective instruction have a number of common characteristics. They cover fewer topics but do so more deeply (Ma, 1999). Good texts have lessons that are based on a progression of mathematical development. They outline tasks that encourage students to think more deeply about a given topic, to make connections among topics, and to move towards increasing efficiency and abstraction. In addition, good texts, like good instruction, allow and encourage students to make sense of the mathematics in their own way as well as to use their own methods of recording solutions. Finally, good texts use engaging, thought-provoking contexts for students.

Good texts come with guides that support the teacher's own knowledge of mathematical content and methods of instruction (Ball & Cohen, 1996). This is crucial, because teachers may want to use the same lesson but in very different ways, depending on their own understanding of the purpose of the lesson. Such guides also give teachers background information on the various student strategies they might see and how these fit into an overall continuum of mathematical development. Finally, textbooks and guides support a variety of methods for assessment, with a focus on observation, interview, and conferencing.

Many other teacher resource books meet many of the criteria of good texts but do so without an accompanying student text. These books or "replacement units" can often offer the detailed and careful development of one concept that may be beyond the scope of a text, which must cover all strands. Teachers should have access to these books, since one textbook is not sufficient to implement all aspects of an effective mathematics program. A recommended list of these books can be found in Appendix B, in the section entitled "Specific Content Area Resources and Units".

Technology

Technology that is used well can play an important role in the junior classroom. Exploring mathematics with technological applications should be an integral part of the junior mathematics program. Many junior students use technology on a daily basis to investigate and to communicate ideas. Technology is part of their world and the world of their future (deSessa, 2000). Technology is not meant to replace mathematical thought but to expand it. Mathematicians themselves use technology. In mathematical activities, tools not only expand cognitive capabilities, they transform them (Wersch, 1985), and computer tools have become as fundamental to the work of professional mathematicians as the protractor was to the work of the ancient geometers.

In the junior years, technology can be incorporated to support the development of student understanding in the following ways:

- It can support students' sense making by providing an environment for investigations. For example:
  - Calculators can be used to look at number patterns and to develop number sense.
  - Spreadsheets can be used to organize and display data, and also to study number patterns and shapes.
  - Online applets can be used at home or in the classroom. They allow students to carry out repeated experiments and to explore visual mathematical relationships.
- It can allow students to develop critical-thinking skills through analysing and comparing many examples.
- It can support the development of diagrammatic reasoning by providing visual representations.

Technology changes the mathematics that students do and the way that students do mathematics. It changes teachers' priorities about what needs to be taught. For example, in the twenty-first century it is not necessary for students to do extensive calculations by hand, but it is necessary for them to develop deep number sense (Reys & Arbaugh, 2001). Technology also allows students to "play" with ideas – both numerical and geometrical – and to ask questions and make hypotheses about their world that they could not before. Through self-directed discovery, students gain access to new levels of mathematics (Sinclair, 2004) and develop their curiosity and a willingness to consider various options. This suggests that during the junior years, experiences with technology should include opportunities for free exploration.

At the same time, the successful use of technology in a situation does not necessarily guarantee that expertise will be transferred to another medium; for example, students who are able to create graphs on the computer will not necessarily understand how to use them in other contexts. Teachers must provide opportunities for students to link the actions carried out in one environment to the other. In fact, "research shows that computer activities yield the best results when coupled with suitable off-computer activities" (Clements & Sarama, 2002, p. 342).

Calculators warrant some additional discussion. Research findings to date indicate that unrestricted access to calculators does not adversely affect student performance in mathematics (Ruthven, 1999). Teachers can help students move towards the appropriate use of technology by:

- including questions, activities, and investigations that use calculators to support realistic problem solving. For example, calculators "can enable pupils to tackle a problem using direct strategies which call for computations beyond their current capabilities; and can also support indirect strategies based on trialling or building up towards a solution" (Ruthven, 1999, pp. 203–204).

- having technology readily available, so that students move towards using it in a natural way (e.g., use calculators as required to carry out complex calculations);
- using both calculators and other technology in paired work to encourage the communication and development of mathematical ideas;
- restricting access to calculators only in specific instances when the main aim is to help students hone mental and written computational skills.

In particular we recommend the use of a two-line display calculator that allows students to keep track of their earlier entries.

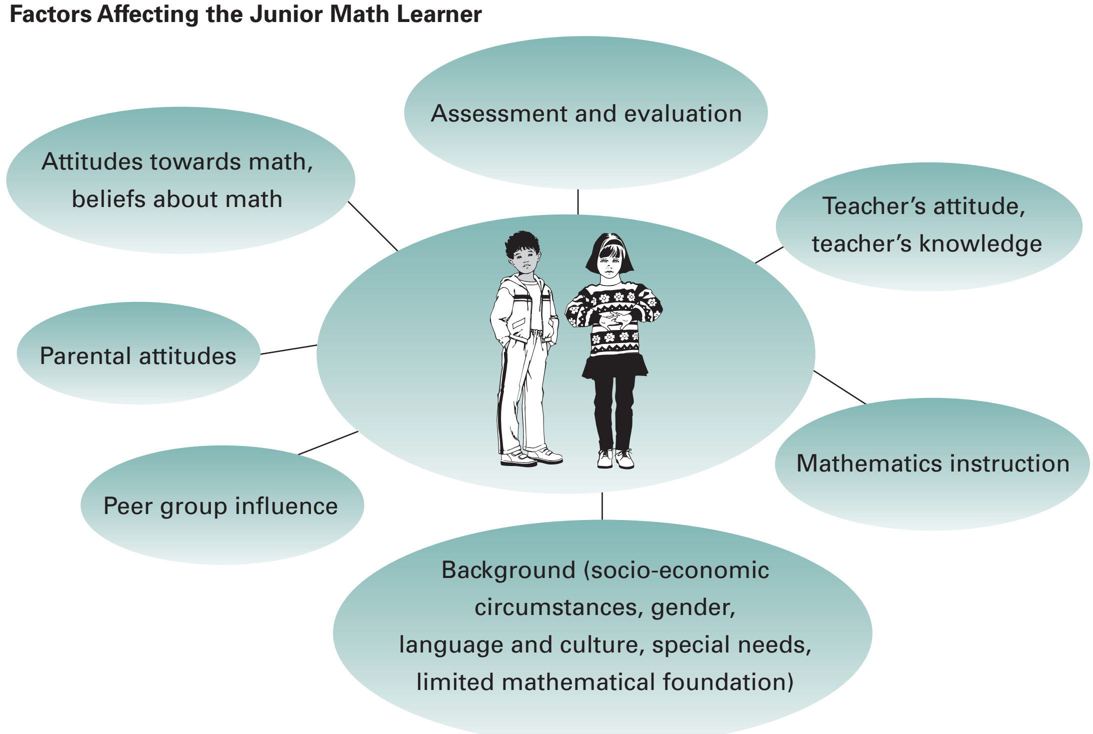
Resources in Ontario French-Language Schools

It should be noted that the limited number of French-language text, literature, and software resources in junior mathematics poses a particular challenge to the French-language school system. Teachers in that system must rely more heavily on their knowledge of the topic and the pedagogy to meet the needs of their students and to be effective in the classroom. It would not be realistic to try to match for the French-language system every learning resource that the marketplace makes available to the English-language system. The ministry should aim, however, to provide sufficient financial support, in addition to what has been traditionally available to French-language schools, for the development of a limited number of high-quality junior mathematics learning resources and for sustained teacher professional development to accompany those resources.

3 Other Factors Affecting the Junior Math Learner

Instruction plays a central role in the junior student's learning and understanding of mathematics. However, it is not the only factor affecting the student's level of mathematical literacy. The junior student enters the classroom not as a blank slate but as a maturing student with an extensive experience of mathematics both inside and outside the classroom. Students have their own ways of viewing school and mathematics that derive in part from their identities as friends or students, and family or community members. What a student brings to the classroom – cultural background, gender, family background – plays a role in how that student experiences and learns in school generally and in mathematics specifically. In the graphic below, some of the factors that affect the junior mathematics learner are identified. In this chapter, we will consider those factors that can affect the student's development of mathematical literacy quite apart from instruction.

Factors Affecting the Junior Math Learner



Attitudes Towards Math and Beliefs About Math

The junior years have a significant impact on whether students see themselves as capable of mathematics as well as on whether they view mathematics as an interesting subject worth pursuing. A student's attitude towards mathematics and capacity for mathematics are inextricably linked; each affects the other (Middleton & Spanias, 1999). In their review of two decades of literature on motivation and mathematical achievement Middleton and Spanias found that "motivations toward mathematics are developed early, are highly stable over time and are greatly influenced by teacher actions and attitudes" (p. 80). During the junior years students can develop a narrow view of mathematics as a set of procedures to be learned and memorized. For some students, math is viewed as "…usually strict. You have to do this, this way, because everybody's already found out the right way" (Gadanidis & Schindler, in press). However, narrow beliefs like this need not prevail.

Effective instruction that is focused on problem solving, a range of solution methods, and student communication of ideas often results in a more positive attitude towards mathematics, a less narrow view of the subject, and a stronger understanding of mathematics than is the case when instruction is focused on the transmission of rules and procedures (Cobb et al., 1991; Wood & Sellers, 1997). When teachers are able to create a genuine community of mathematics learners – a risk-free environment in which discussion is encouraged and students feel comfortable sharing their ideas – then students are more likely to find that mathematics can be an interesting and enjoyable activity. Their views about mathematics and what it means to do mathematics will change.

Background

All students deserve to become mathematically literate regardless of gender, socio-economic background, language, cultural background, learning ability, or previous mathematics experiences. Instruction must address the needs of students from a wide range of backgrounds. As Van de Walle and Folk (2005) assert, "It is no longer reasonable to talk about the 'regular classroom.' It is even more difficult to talk about the 'average child'" (p. 458). The type of instruction described in this document is designed to allow teachers to take into account these differences and value who the student is and what the student's background and previous knowledge are. It should be recognized, however, that certain situations require either a modification of the methods we have outlined or more than what is outlined. Each of these situations is addressed in the following pages.

All students benefit from good instructional methods, which can be instrumental in helping them overcome real or perceived disadvantages that they may have as a result of:

- living in impoverished circumstances;
- having a different language or different cultural background;
- having special needs.

Socio-economic Circumstances

Teachers have been successful using problem-oriented methods in schools in all types of neighbourhoods, regardless of the socio-economic circumstances of the students. For example, in her work comparing "problem-oriented" with traditional "procedural" instruction in two low-income neighbourhood schools, Boaler (2002) found that, after three years in the problem-oriented school, students obtained a significantly higher level of achievement on a range of assessments, including the national examination, at age 13. Furthermore, she found more equitable gender achievement. Boys in the procedurally oriented school obtained significantly higher grades than the girls did, whereas there were no gender disparities in grades at the problem-oriented school – all students performed better on average. We do not wish to diminish the challenge that problem-oriented instructional practices may place on a teacher in a particular classroom, or with particular students; rather, we stress that, despite obstacles, these methods have demonstrated strong results in all types of classrooms and should be made available to all students.

Gender

There has been extensive research on the topic of gender and mathematics. On the one hand, Sanders and Peterson (1999) summarize the research regarding girls' mathematical achievement by stating: "What was once an alarming gender gap in math achievement and participation has been reduced to a few percentage points . . ." (p. 47). In school achievement girls now typically fare as well as boys in mathematics – a dramatic change from earlier times. However, for students who go on to university there remains a discrepancy, and a growing one at each level in mathematics, in favour of men (Burton, 2004). Whether or not this discrepancy has roots in the junior grades is speculative, but it cannot be entirely ignored.

On the other hand, there is also a growing concern that in fact boys are not faring as well in schools as they might (Ravitch, 1998). Boys are overrepresented in schools in behavioural classes, learning disability classes, and special needs of all sorts (Lajoie, 2003). The contention from some groups is that boys' stereotypical behaviour is not necessarily conducive to learning through traditional modes of instruction (e.g., sitting quietly for long periods). We feel that it is important to be aware that there are differences in how students do, understand, and think about mathematics. The methods that have been suggested in this document begin with this premise, which – while not a panacea – acknowledges the differences that students bring to the classroom. It may be, however, that teachers must go beyond this, and also pay careful attention to the context of problems: Do they engage all the students? Do students of both genders view mathematical inquiry as a positive and interesting activity?

Language and Culture

Teachers can make effective use of problem-oriented methods with specific modifications to better address the needs of students from different cultures or languages. Instruction that includes a student's culture in different ways is more likely to engage the student. In one example, students began with narratives of their home experiences in their own language as a basis for mathematics in the classroom (Lo Cicero, De La Cruz, & Fuson, 1999). The teachers and students used student stories and pictures from home to build math problems that related to the students' everyday lives but served equally to advance their mathematical knowledge.

Learning mathematics in a second language (whether as an ESL student or an English-speaking French immersion student) can initially add a layer of difficulty in mathematics. The same techniques used to support literacy skills can be used in the development of mathematics – for example, working where possible in both languages, making use of manipulatives and diagrams to communicate, working with a fellow student in the same original language. In the case of students who have withdrawal services for language development, having them write mathematics word problems (to share with other students) can serve as a link between the instructional experience during withdrawal and that of the regular classroom.

Bresser (2003), in his work with ESL students communicating mathematical ideas, suggests that teachers:

- allow wait time for students' responses;
- pose problems in familiar contexts;
- connect symbols with words;
- have students share with their partner first, then with the whole class;
- use "English experts" (students with the same native language but a stronger grasp of English);
- have students "retell" another student's explanation.

In French-language schools, the linguistic support programs Actualisation linguistique en français and Perfectionnement du français (see the curriculum policy document for ALF/PDF, 2002) play an important role in ensuring that students acquire the necessary level of skills in the French language in order to fully understand the mathematics concepts, share their ideas, and communicate their thinking. The mathematics learning and teaching approaches put forward in this report emphasize the communication and sharing of mathematical thinking in a safe and respectful environment. The Ministry of Education is encouraged to develop support materials that would facilitate the implementation of the ALF/PDF curriculum guidelines, particularly as they apply to mathematics in the primary and junior grades.

Special Needs

Students with learning disabilities will benefit from problem-oriented instructional methods. Students with learning disabilities are often relegated to a narrow instruction in mathematics as simple rules and procedures (Woodward & Montague, 2002). Special education typically places considerable emphasis on rote learning and mastery of math facts and algorithms for basic operations rather than on problem solving.

Narrow procedural knowledge will not be sufficient for students with special needs in their future schooling or work. Like all students, students with learning disabilities will benefit from a problem-solving approach focused on making sense of mathematics. Beyond this approach, students with specific disabilities may need additional attention. Sliva (2004) presents different types of learning problems and how they may affect mathematical development; her observations are summarized in Figure 7 on pages 36 and 37.

For students with special needs, additional support in some foundational math concepts is often necessary, and recommended, as the mathematics they are expected to learn becomes increasingly more complex through the junior grades. Instruction needs to begin with what students understand. Teachers may need to make use of primary mathematics concepts, such as place-value understanding and counting principles, which are discussed in the series of guides to effective instruction in mathematics for Kindergarten to Grade 3 published by the Ministry of Education. When teachers are able to recognize and identify the gaps in students' understanding, they can choose appropriate activities and problems to help close those gaps.

Students who have needs that cannot be addressed adequately in the regular program should have access to additional support in mathematics, including the support of special education teachers with specific training in mathematics education. The mathematics instructional training provided for these teachers should include an examination of some of the big ideas in primary mathematics that students must understand in order to progress – for example, the decomposition and recomposition of numbers.

While the field of addressing learning disabilities within a problem-solving framework is relatively new, programs that are being developed are worth investigating. See, for example, the primary division program Mathematics Recovery (Wright, Martland, Stafford, & Stanger, 2002).

Special consideration should be given to all students with special needs, including gifted students. Gifted students also need to be provided with interesting, rich, and challenging programs. Using a balanced approach to instruction, and teaching math concepts through problem solving, teachers can provide the students in their classes with problems that have entry points for all learners. Problems should be sufficiently rich to engage gifted students. Gifted students may require additional extensions or enrichments that will help them further develop their understanding of the mathematical concepts being explored by the class.

Figure 7: Types of Learning Problems and Their Possible Effects on Mathematical Development

Type of Learning Problem	Possible Skill Areas Affected
Spatial perception	– dealing with the directional aspects of mathematics (e.g., solving problems involving single-digit addition [up-down], regrouping [left-right], aligning numbers, or using a number line) (Miller & Mercer, 1997) – understanding the concept of fractions, writing fractions, writing decimals, discerning differences in size and shape
Reversals	– regrouping and transposing digits
Figure-ground perception	– maintaining the sense of a problem and not mixing up parts of different problems – reading a calculator – reading multidigit numbers – copying symbols properly
Visual discrimination	– identifying symbols – gaining information from pictures, charts, or graphs – being able to use visually presented material in a productive way – telling the difference between a quarter and a nickel, the numbers 6 and 9, and the small hand on a clock and the large one – using many mathematics skills (e.g., in measurement, estimation, problem solving, and geometry)
Auditory processing	– hearing a pattern in counting – deciphering numbers that are spoken (e.g., 30 and 13) – performing oral drills – identifying ordinal numbers
Motor ability	– presenting legible work (However, the teacher must not infer that a messy paper indicates that the student does not know his or her mathematics.)

Type of Learning Problem	Possible Skill Areas Affected
Expressive ability	– communicating about mathematics, both in writing and orally
Receptive ability	– connecting vocabulary words with an understanding of mathematical concepts (e.g., *first, greater than*) – understanding words with multiple meanings (e.g., *sum, times, difference*) – following directions and solving word problems – ignoring irrelevant numerical and linguistic information in word problems
Cognitive and metacognitive abilities	– selecting an appropriate strategy – being aware of basic skills, strategies, and resources necessary to complete mathematical tasks – organizing information – monitoring problem-solving processes – evaluating problems for accuracy – generalizing strategies to new situations
Attitude	– believing that he or she will be good in mathematics – thinking on one's own and taking risks – believing that he or she will be successful at learning important mathematics concepts

Limited Mathematical Foundation

In addition to students identified as having learning disabilities, there may be other students, harder to identify, who are struggling quietly because they do not have a sufficient understanding of fundamental mathematical concepts. Teachers who focus on the oral communication and explanation of ideas can make use of their discussions with students, along with individual interviews, to identify struggling students and prevent them from "falling through the cracks". Such students will benefit from a program based on solving problems with a range of entry points and solutions that can be reached through a variety of strategies. When teachers use paired or group work, they may want to place these students with other students at a similar level of understanding. Students in such pairings will have an opportunity to explain, test out, and develop their own ideas, rather than being shown and told what to do by a stronger student.

Peer Group Influence

The peer group plays an increasingly important role in students' attitudes towards school generally, including their attitude towards mathematics. As students move into middle childhood and the beginning of adolescence, peer groups with distinct norms and social structures emerge; students often prefer the company of their peers to that of adults. To acquire acceptance into a peer group requires a certain amount of social approval, and this social approval is contingent upon conformity to the norms of the group. Depending on the group, this peer pressure can become positive or negative, either improving students' participation in school or reinforcing negative social behaviours such as misconduct in class. Students who feel socially accepted have higher self-esteem, and higher self-esteem is correlated with stronger academic performance than low self-esteem (Birch & Ladd, 1997). Establishing a classroom community where positive peer influence is encouraged and expected is important for the learning in this age group. Valuing the contributions of all students and expecting students to respect others and themselves help to develop positive peer interactions that in turn promote positive peer influence.

Parental Attitudes

Parents should be included in their children's mathematics education in a meaningful way. In her review of the research on parental involvement in education in Britain, Merttens (1999) asserts that "whatever we do inside the school gates will never be as effective as it should be unless we turn our attention to what happens on the outside!" (p. 79). Indeed, many researchers feel that it is the parents[6] who are the single biggest factor in a child's educational success. Parents foster a positive attitude towards mathematics in their children by demonstrating an interest in math, modelling perseverance in problem solving, and highlighting mathematics as it is encountered in their workplace and in the home environment. A positive attitude on the part of the parents supports the teacher's classroom efforts.

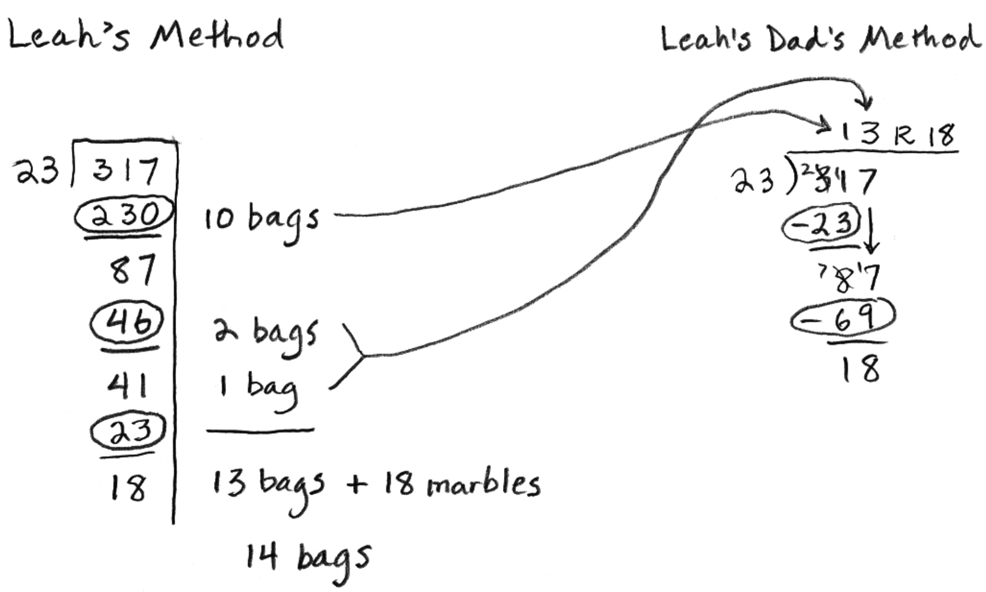
Including parents meaningfully means making them feel comfortable and welcome in their child's school. This may be particularly important in the case of mathematics, given that many parents' experience of mathematics in school may have been less than positive and that they are now faced with new mathematical content (e.g., probability) taught in unfamiliar ways.

6. *Parents* refers to both parents and guardians.

There are effective ways that teachers and schools can forge strong links with parents that benefit all involved. Many teachers already make good use of games and interesting problems for homework and optional home activities. In addition, as students share at home the problems that they are working on in school, parents will have their own ways of solving these problems, and these ways can be included in school discussions. For example, Ms. H addresses this issue in her classroom.

After working on the marbles problem Ms. H had three students report that they had learned another way to solve the problem at home. As one student, confided, "I showed my dad my way and he showed me his way. I didn't understand his and he didn't understand mine!" Ms. H capitalized on this confidence to look at what was similar between Leah's way and her dad's way of solving a division problem. Some of the students were able to find the link between the two methods (shown below). Other students were unsure. Ms. H pursued the comparison because she wanted to make sure that a connection was made between what was happening in her class and what parents were sharing at home. She did not want a disjunction between math at school and math at home; instead, she wanted one to strengthen the understanding of the other.

A Comparison of Leah's Division Algorithm and Her Father's "Standard" Algorithm



Homework tasks must provide meaningful experiences for both children and parents. Activities should be engaging mathematical experiences that highlight mathematics in the students' environment. Whether students are playing a game requiring strategy with family members or friends or completing an assigned task, their mathematics homework should be meaningful, enjoyable, and productive.

4 Mathematics Assessment and Evaluation

Assessment and Learning

Effective assessment and evaluation practices have the capacity to greatly strengthen a student's mathematical literacy. Although assessment and evaluation are used for a range of reasons, their fundamental purpose – and the purpose against which all other purposes should be weighed – is to support students' learning and understanding of mathematics (Wilson & Kenney, 2003). As the National Council of Supervisors of Mathematics (NCSM) contends: "In order to develop mathematical proficiency in *all* students, assessment needs to support the continued mathematical learning in *each* student" (1997, p. 1-11). This is an achievable goal.

Well-constructed and well-implemented assessment plays an essential role in the improvement of student learning. Black and Wiliam (1998) found in their review of studies on assessment and student achievement that in classrooms where teachers used formative assessment (i.e., assessment as an ongoing part of learning), students achieved at significantly higher levels than in classrooms where this was not a feature of instruction. They concluded that the achievement gains were "larger than most of those found for education interventions" (p. 141). The gains came from classes with effective assessment practices, which will be discussed in this section. Black and Wiliam also distinguish between assessment and evaluation. Assessment is an ongoing examination of what students know and can do, while evaluation is the interpretation of assessment data and, if required, the assignment of a grade. Although Black and Wiliam found that good assessment was linked with improved learning, this was not the case in classrooms where there was an overemphasis on evaluation in terms of grades. Where the emphasis was on evaluation, students focused on obtaining the grades, often by superficial means, rather than by focusing on learning deeply.

Effective assessment and effective instruction are not necessarily different activities and in fact should become nearly indistinguishable (Stenmark & Bush, 2001). Assessment is an ongoing part of the learning-teaching process and includes regular opportunities for students to demonstrate their learning (Expert Panel on Early Math in Ontario, 2003). As Fosnot and Dolk (2001b) suggest, "[A]ssessment must be dynamic in that it evaluates *movement* – the journey. But it must also be dynamic *by being directly connected to learning and teaching*" (p. 129).

PURPOSES OF ASSESSMENT

The teacher undertakes the assessment process by first determining the purposes of the assessment, which are to:

- determine the students' prior knowledge;
- know what students have learned on a given topic;
- make decisions about future lessons;
- identify individual difficulties;
- obtain information for discussion with the student, their parents, and the administration.

Teachers use assessment for diagnostic purposes, to determine what students know and therefore what they need to learn. Teachers also use assessment to understand what students have learned, for the purpose of evaluating their own lessons, determining where the next lessons should go, and later being able to report on learning. Teachers assess student learning to provide information on the class as a whole and on individual students. The whole-class information allows teachers to know what the students in general are learning or have difficulty learning and therefore helps guide the direction of future lessons. This is ongoing or formative assessment as a unit progresses. The teacher is also examining individual assessments to determine what individuals know and need to learn. We do not think that it is useful, or indeed possible, to have any instrument (e.g., a diagnostic developmental continuum[7] or test) that would tell teachers that, on the basis of a given sample of work, a student is at an exact stage of learning (Wilson & Kenney, 2003). While such continuums are extremely useful and are one key to informed assessment and instruction, they are only part of the picture the teacher will use. A continuum cannot "know" a student: that is the job of the teacher. A teacher's observations of students in problem-solving situations will reveal considerably more about the students' understanding of math concepts, their sense making in math, and their individual strengths and weaknesses.

METHODS OF GATHERING INFORMATION FOR ASSESSMENT

In general, assessments should be informative, non-intrusive experiences and should not make students *feel less than they are*. Assessments should encourage students to show what they know and can do, rather than focus on what they do not know or cannot do. We agree with the recurring call over the last five decades for increasing the use of more informal methods of assessment – observation (of oral and written work) and discussion

7. A diagnostic developmental continuum is an assessment aid in which students' methods of solving problems are described and organized into a progression. The continuum is accompanied by suggested ways of fostering students' progress in mathematical concept development.

(with the student) – and relying less on formal paper-and-pencil tests (Lambdin, 1993). As students and the teacher communicate about their ideas and strategies, teachers can obtain useful information about student understanding and thinking.

In addition to ongoing informal assessment, teachers can also make use of various pieces of information, including mathematical tasks, a portfolio of work, mathematical projects, and short tests that can be accumulated over time (for a sample of these, see Stenmark & Bush, 2001). In their overview of the research, Wilson and Kenney (2003) summarize the characteristics of good tasks. They should "be novel and varied in interest, offer reasonable challenge, help students develop short-term goals, focus on meaningful aspects of learning, and support the development and use of effective learning strategies" (p. 53).

Finally, teachers can also gain information through short interviews and conferences about the students' work.

Use of Assessment Information

As teachers assess, they gain information about both the individual child and the class as a whole. They then use this information to inform classroom practice and determine where to head in future lessons. This is a significant aspect of effective assessment. Teachers should continually adjust their program in response to the information that they learn through assessment. If assessment data have no impact on teacher action, then assessment will not be linked to student improvement (Black & Wiliam, 1998).

Teachers also make use of assessment information to speak individually with students. Students must be involved in the assessment process, getting feedback from the teacher and taking part in their own goal setting in response (Stenmark, 1991). Including students in their own formative assessment helps them to develop responsibility for their work and promotes their autonomy as learners.

Use of the Achievement Chart

The achievement chart on page 9 of the 1997 Ontario mathematics curriculum policy document offers teachers a way to think of their mathematics program across the categories of Problem Solving, Understanding of Concepts, Application of Mathematical Procedures, and Communication of Required Knowledge. We suggest that teachers do not try to give separate grades for the four categories. These are not separate, measurable parts or "constructs". These are intertwined parts of mathematics that are more accurately assessed as a whole (Wolfe, Childs, & Elgie, 2004). Sometimes the "application of procedures" category is given disproportionate weight among the four categories when a student's work in mathematics is being assessed; the student's work should be assessed on the basis of criteria in all four categories.

How will Ms. H assess her students' understanding of division and multiplication? She has seen how her students solved the problem presented, having looked at their written work, having listened to their discussion over the two-day lesson, and having observed their work in later lessons when they tackled a number of other problems. As students progress throughout this unit, Ms. H. is assessing their discussion and work by asking herself various questions. Do they understand division (when to use it, how to deal with the remainder)? How do they solve the problems (e.g., are they at an earlier stage of multiplying up or a later stage of using an efficient algorithm?) Do they solve the problems accurately? If not, what types of errors do they make? Ms. H. will use all of this to think about what the students understand and where they should be headed in later lessons. After students have done more written work and taken part in further discussion, Ms. H. will ask them to complete a task on their own: "Your task is to write three division problems. First, write a problem you consider to be easy to solve. Next, write a problem you think is of medium difficulty. Finally, write a problem that you feel is hard. Solve all three" (Wickett & Burns, 2003, p. 252). She will use all of this to evaluate their knowledge for the purposes of reporting.

5 Support for Mathematics Education and Learning

Supporting mathematics education and learning is a shared responsibility that encompasses all members of the educational community, including the Ministry of Education, district school boards, principals, lead teachers, teachers, faculties of education, and parents. All partners play a vital role in ensuring that optimal conditions for learning and the necessary resources and professional development are present at all levels.

Professional Development in Mathematics Education

Becoming an effective mathematics teacher can be a complex, rewarding, and yet challenging endeavour, particularly as teachers undertake the kind of instruction described in this report. The effects of this new situation can be summarized as follows:

- Teachers are asked to teach in a way that they themselves may not have experienced and have not likely seen in classroom situations.
- Teachers need a more extensive knowledge of mathematics than they have previously needed as teachers and as students.
- Teachers need to develop deep and flexible knowledge of pedagogy in order to work effectively with students' thinking and alternative strategies.
- Teachers may find it difficult to carve out the recommended time of one hour a day, given the number of curriculum expectations (the required knowledge and skills that students must develop and demonstrate over the course of the year in all subjects), and will require support in incorporating the big ideas into their planning and reporting.
- Some parents, students, and principals may have a more traditional view of what good instruction looks like and may need information on why and how mathematics instruction has changed.

For professional development to be effective, meaningful, and relevant, it must help teachers address each of these issues.

Characteristics of Effective Professional Development

1. Effective professional development is focused on specific goals that are clearly connected to mathematics and mathematics teaching. Professional development should be long-term, with several short-term, realistic, manageable goals in mind. Mathematics teachers need extended experiences of doing mathematics and working with the new methods. Professional development focused on the teaching and learning of specific mathematics content is more effective than more general professional development. In their comparisons of the research on professional development, both Kennedy (1998) and Cohen and Hill (2001) found that teachers who took workshops that were extended in time and focused on examining students' mathematical work reported more effective classroom practice. In contrast, teachers who took workshops more loosely focused on hands-on activities, gender, cooperative learning, and other tangential topics were less likely to report such practices.

2. Effective professional development supports the development of teachers' knowledge of mathematics. Teachers whose understanding of mathematics has been deepened are much more likely to help students make important mathematical connections, to teach to the big ideas, and to see the mathematical value of particular problems. Teachers must understand their subject matter deeply in order to teach effectively. Professional development needs to create mathematics experiences that cause teachers to reflect on their knowledge and beliefs and to see mathematics and mathematics teaching in a new light (Gadanidis, Hoogland, & Hill, 2002). When such moments of epiphany occur, images of mathematics education – such as curriculum documents, classroom experiences, ideas from professional development workshops, journal articles, and so forth – shift and something new is seen, something that was not apparent before. As one teacher commented, "I feel like [this experience] has cleaned my spectacles and I am reading the [curriculum] document with new vision" (p. 1612).

3. Effective professional development supports the development of teachers' knowledge of how children learn mathematics (Garet, Porter, Desimone, Birman, & Yoon, 2001). Teachers may find it a challenge to implement some recommendations for effective instruction when they have limited information and training about how children best learn mathematics, how students might solve problems, what their typical strategies might be, and what their thinking is. When teachers have the opportunity to learn about children's mathematical development, they respond more effectively to the mathematical needs of their students, know how to work with student ideas, and make the most of a given math moment. This professional development should include either video data or actual classroom settings wherein teachers can see, as well as read about, students' mathematical development in problem-based programs.

4. Effective professional development is active learning – it gives teachers the opportunity to try new ideas and discuss them. Effective professional development offers teachers ongoing opportunities to try new ideas in their classrooms and to share and discuss their teaching and their reflections on their teaching in a professional, collaborative community. The purpose of the professional development of teachers is to improve student learning and understanding of mathematics. For teachers, this means trying out new strategies with their students. For significant change to occur in teachers' beliefs and practice, teachers need experiences in which they engage in practical inquiry and reflection about mathematics and mathematics teaching (McGowen & Davis, 2001; Stipek, Givvin, Salmon, & McGyvers, 2001). Discussion among teachers who teach the same grade and share many experiences and issues can help teachers make sense of their experiences and feel less isolated. This analysis of and reflection on their practice may take the form of talking with others, keeping a journal, engaging in action research (Darling-Hammond & Ball, 2000), or engaging in collaborative research (Bednarz, 2000).

5. Effective professional development includes support from knowledgeable others. While teachers can benefit from school-based professional development, this model is often more effective when stimulated by an elementary mathematics consultant/coordinator or resource person. The leader should have the specialized knowledge of mathematics teaching and learning and the experience in the elementary context to help to promote effective development. The leadership and input from a mathematics consultant/coordinator or resource person is particularly important in mathematics, an area in which many teachers and administrators may feel uncomfortable and may lack deep knowledge of both content and pedagogy.

6. Effective professional development values teachers as professionals. Teachers are professionals who are most effective when they are treated as such. Effective professional development must have a range of entry points for teachers who have varied needs. The professional development models should build on what teachers already know. Teachers should be able to choose the professional development that is most appropriate for their own situation and their own classroom context. Teachers need to be trusted in the same way that other professionals, such as lawyers, doctors, and architects, are trusted (Fullan & Connelly, 1987).

These characteristics are summarized in the following list:

Characteristics of Effective Professional Development

Effective professional development:

- is focused on specific goals that are clearly connected to mathematics and mathematics teaching;
- supports the development of teachers' knowledge of mathematics;
- supports the development of teachers' knowledge of how children learn mathematics;
- is active learning – it gives teachers the opportunity to try new ideas and discuss them;
- includes support from knowledgeable others;
- values teachers as professionals.

Routes to Effective Professional Development for Teachers in Ontario

There are a variety of different routes that teachers can consider to strengthen their instructional capacity in mathematics. These include initiatives at the district, school, university, and professional levels. The most effective professional development routes are those that meet the characteristics outlined in the list on page 47.

Ministry and Board-wide Initiatives

Teachers can take advantage of a range of professional development activities offered by the ministry and by their boards. During the 2003–04 school year, the ministry sponsored high-quality, intensive professional development for lead teachers in Kindergarten to Grade 3 mathematics. Some boards made this same training available to all their Kindergarten to Grade 3 teachers. A version of the training was also offered free to all elementary teachers, Kindergarten to Grade 3, in the ministry's Summer Program 2004.

A similar program of intensive professional development is expected to begin in the 2004–05 school year for lead math teachers in the junior level. Boards should attempt to extend the training to as many junior grades teachers as possible, and teachers can take advantage of offerings for professional development in future Summer Programs.

School-Based Initiatives

Schools offer various ongoing in-service projects for teachers, in particular, the development of collaborative teams of teachers working either on their own or with an outside expert.

Professional Association and Federation Initiatives

Some professional associations – for example, the Ontario Association of Mathematics Education (OAME), the Ontario Mathematics Coordinators Association (OMCA), and the National Council of Teachers of Mathematics (NCTM) – and teacher federations offer a range of workshops, courses, and resources in elementary mathematics. Some teacher federations provide additional support for teachers attending courses offered throughout Ontario and other jurisdictions. The OAME also offers conferences regionally and provincially and other forums for professional development.

University-Based Courses

At the university level teachers can enrol in:

- Additional Qualifications courses in primary/junior mathematics. Because these courses involve a significant investment in time and monies, some boards have financially supported teachers to take them. In addition, some boards have worked with universities that are offering these courses to tailor the courses to the needs of their teachers. The lack of access to such courses is a particular concern for the French-language community – no Additional Qualifications courses in mathematics are available in the French language.

- A masters of education degree with a focus on mathematics education. Enrolment offers teachers an opportunity to carry out research or portfolio development in the area of mathematics.

Joint Board and University Initiatives

Teachers can take part in professional development research projects created through agreements between local universities or researchers and specific schools or boards of education. Typically these projects are part of ongoing research initiatives originating in the universities. The projects include teacher in-service and research on instruction and learning within their classrooms. These projects could be expanded upon and further viewed as another route to accreditation and continuing education for teachers.

Developing Capacity for Mathematics Improvement in Ontario: Responsibilities

Professional development is the key to strong mathematics instruction. The instructional practices outlined in this document will not happen without sufficient support in the form of effective professional development for teachers. The requirements of professional development may be greater in mathematics than in other subject areas at the junior level because of the dual necessity of developing a strong knowledge of mathematical content and developing new instructional methods.

Effective professional development is a sustained commitment by all members of the system. If professional development is to support teachers, it must be sustained. Short-term professional development is not enough to support teachers who are undertaking the type of instructional changes outlined in this document. It must happen regularly and must take place over extended periods of time (Glickman, 2002). Since teachers are to try out new ideas and strategies, they may experience moments of celebration and also frustration. Therefore they need to have ongoing support that provides a forum for analysing, discussing, and considering how these ideas and strategies work in the classroom. Genuine improvement in mathematics instruction takes time and support.

Effective professional development is valued and supported by school and school district administrators. Senior administrators and principals are the key to creating the conditions for the continuous professional development of teachers and, thus, for classroom and school improvement (Fullan, 2001, pp. 137–150). Principals and other administrators need to be actively involved in the professional development process and should make informed decisions about professional development at the board and school levels (Burch & Spillane, 2001; Payne & Wolfson, 2000). Principals and senior administrators also need professional training in what sound early mathematics experiences should look like; this training should be specifically tailored to the needs of principals and other leaders in the system. An effective professional development program that is designed to build awareness and support for mathematics initiatives includes development for principals and senior administrators.

The following section outlines roles for the various education partners and introduces a new role, mathematics facilitator, which is recommended by this panel.

Ontario Board Leaders

Success in mathematics must be an identified goal in all school boards across Ontario. Board leaders play an important role in articulating a clear vision and priority regarding mathematics education across their school districts. These board leaders provide leadership in:

– creating a vision and focus for mathematics;

– fostering leadership within;

– allocating resources.

Creating a Vision and Focus for Mathematics

A key role of the board leader is to promote and establish a shared sense of vision and purpose within the school district. In this role, board leaders:

- facilitate a system-wide commitment and priority to mathematics education;
- communicate a shared vision of mathematics instruction across the board and assist principals in developing a shared vision within their schools;
- work to minimize competing priorities in order to establish a focus on mathematics;
- ensure the alignment of board goals with provincial-, school-, and classroom-level goals.

Fostering Leadership

Strengthened leadership capacity is a prerequisite for successful innovation. Establishing leaders in mathematics education is essential if professional development is to be implemented and sustained and teacher capacity and students' mathematical literacy are to be increased.

Board leaders:

- establish board-level mathematics program support, with personnel who have extensive knowledge and experience in elementary mathematics;
- establish the role of mathematics facilitator (described later) within a family of schools to support administrators and lead teachers;

- work with principals to clarify the vision of effective instruction in mathematics across the system;
- emphasize connections between instructional strategies and programming found in the recent Expert Panel reports and related resources on primary and intermediate mathematics to enhance the cohesiveness of the instructional strategies and programming applied in junior mathematics;
- ensure the availability of professional development tailored for principals, to help them learn about the components of mathematics instruction and to facilitate mathematical learning in their schools.

Allocating Resources

Effective allocation of resources provides administrators and teachers with access to the tools they need to enhance instruction and student achievement.

Board leaders:

- supervise the acquisition and development of resources to support mathematics instruction and learning;
- provide flexibility to compensate for individual school needs;
- allocate financial resources according to identified priorities;
- support mathematics personnel to assist with implementing mathematics initiatives at the board level and among families of schools.

Principals

Knowledgeable principals are the key to creating the conditions within a school that help to promote mathematical learning and achievement. Principals work to build a community of learning within their school and strive to ensure that teachers are fully supported as they implement their mathematics program. Principals provide leadership by:

– supporting classroom instruction;
– building a collaborative team;
– providing resources and support;
– promoting home and school partnerships.

Supporting Classroom Instruction

Teachers need to be supported as they begin to implement new ideas and to learn how mathematical concepts develop throughout the junior grades. In an effort to support mathematics instruction, principals should:

- embrace opportunities to increase their knowledge of instructional methods in order to support staff in their journey;

- act as an instructional leader in the school and model the characteristics of a mathematics leader and learner;
- ensure that timetables are organized to provide daily one-hour blocks of time for mathematics instruction;
- stay abreast of new strategies and be cognizant of how the big ideas of mathematics develop throughout the junior grades;
- facilitate opportunities for staff to discuss mathematics within the school day (e.g., through creative timetabling, blocks of planning time);
- monitor teachers' mathematics programs to ensure that components are effectively implemented and observed in classrooms (e.g., teaching through problem solving, integration of communication, inclusion of alternative strategies);
- encourage and support attendance at mathematics professional development sessions outside the school;
- ensure that school-based professional development is focused on mathematical content knowledge and knowledge of pedagogy;
- provide guidance and meaningful feedback about mathematics instruction.

Building a Collaborative Team

Principals play an important role in establishing a culture of collaboration and sharing within their school. In mathematics, a subject that many teachers are anxious about, it is essential that principals establish a sense of community and respect within the staff. In building a collaborative team, principals:

- are aware of the issues related to math anxiety and integrate strategies that encourage teachers to share their ideas in a risk-free environment;
- establish an atmosphere of trust;
- promote the sharing of best practices within the staff;
- encourage learning activities such as book clubs and study groups;
- celebrate the successes of individuals and teams.

Providing Resources

If teachers are to implement new ideas in their mathematics programs, they will need appropriate resources. The shift from traditional instruction to a more problem-based program should be supported by the availability of professional resources, manipulatives, calculators, and computer software. To assist teachers in implementing new strategies, principals:

- ensure funding for professional resources that provide background knowledge and ideas to be integrated into their classrooms;

- provide additional funding so that all teachers have access to manipulatives on a daily basis;
- share information with staff about board-wide and outside-of-board professional development;
- support professional development by allocating funds for staff to attend learning opportunities outside the school (e.g., the OAME conference).

Promoting Home and School Partnerships

Communication with parents about mathematics education is essential. Since many parents experienced instruction that was quite different from the methods outlined in this report, principals play a key role in:

- sharing the school's focus on mathematics with parents and the community;
- including in school newsletters math sections that provide ideas for parents to use in assisting their child with mathematics at home;
- providing background information to parents regarding the rationale for changing instruction in mathematics;
- facilitating family math evenings and information sessions to increase parental awareness.

Elementary Mathematics Support Personnel

Bringing about change in mathematics instruction in the junior division will require the support of personnel in all school boards whose responsibility is to oversee the direction of mathematics at the board level. We recommend that boards maintain or initiate the role of elementary mathematics consultant/coordinator as well as investigate the introduction of a new role, that of mathematics facilitator, who will work with a family of schools under the guidance of a consultant/coordinator.

Consultant/Coordinator

The role of consultant/coordinator is essential to the success of this initiative. Because many administrators and lead teachers lack content knowledge of mathematics and knowledge of pedagogy in mathematics, there is a need for support personnel (consultants/coordinators) who have experience in the elementary panel and have extensive knowledge of mathematics teaching. The shift from the workshops of the past, which focused on generic processes (e.g., keeping math journals), to more content-specific in-services has resulted in an increased need for such positions. The Expert Panel report on primary mathematics (2003), along with the extensive training and resource materials associated with it, has heightened awareness of the need to have elementary mathematics leaders.

In light of the increasing complexity of mathematics concepts in the junior division, there is a need for consultants/coordinators who have training in elementary mathematics and who have a main focus of mathematics. It is a challenge for a generalist consultant to provide in-depth content-based training in mathematics education. Consultants/coordinators must themselves be continually learning in order to stay abreast of current research and best practices in mathematics education. They benefit from attending professional development opportunities targeted to their role (e.g., OMCA, NCSM).

Consultants/coordinators play a vital role in working with senior administrators, principals, and teachers to facilitate mathematical learning throughout a district school board. They also provide guidance and support to our recommended mathematics facilitators and promote linkages between the primary, junior, and intermediate math initiatives.

Mathematics Facilitators

Strong learning and growth have occurred for many primary lead mathematics teachers as a result of the six days of training that were provided during the 2003–04 school year. We want to sustain the work that has already been undertaken in the primary division and build on it into the junior division. As discussed earlier, sustained support for improvement is an essential element of effective professional development. Therefore, we recommend that additional support be offered by the creation of **mathematics facilitators** in each board to support the ongoing learning of both primary and junior mathematics teachers (see the graphic on the opposite page). These facilitators would support lead mathematics teachers within a cluster of schools. They would be individuals who have demonstrated their expertise in mathematics teaching, who are willing to continue their learning through additional training, and who are prepared to assist their colleagues to deepen their knowledge and skills in math instruction.

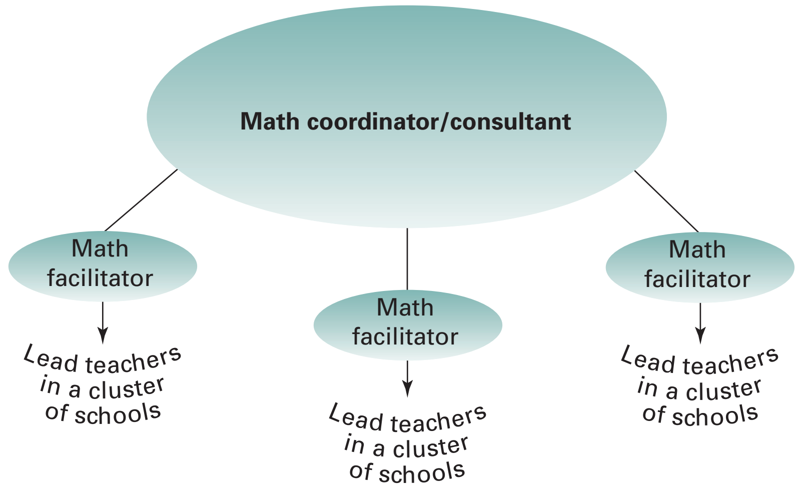
Math facilitators will require sufficient professional development at the beginning of their mandate to reinforce the integration of research-based strategies into their practice. They will also need to explore strategies to help them be successful in collaborating and sharing with other colleagues.

Wherever possible, mathematics facilitators should continue to practise in their own mathematics classrooms. Some boards are now experimenting with this kind of situation, in which a mathematics facilitator continues to instruct in mathematics part-time in one school. It is important to continue to build a critical mass of mathematics expertise in the junior division in Ontario. This model, which includes some teaching, would support that development.

Lead Teachers

The role of lead teacher has recently been introduced in Ontario through the Early Math Strategy, Kindergarten to Grade 3. The first phase of training provided for early math lead teachers focused on the big ideas of one strand of mathematics (Number Sense and

Sustainable Professional Development Model in Mathematics



Numeration in the English-language boards and Geometry and Spatial Sense in the French-language boards) and provided background mathematical knowledge as well as instructional ideas that lead teachers could incorporate into their lessons. In the initial phase, early math lead teachers were not expected to share their learning formally with other teachers at their school, although some did so in an informal way throughout the year.

We anticipate the introduction of junior math lead teachers in the upcoming school year. Lead teachers will help to promote mathematical learning and sharing throughout the junior division in all schools in Ontario. Like early math lead teachers, junior math lead teachers will need much training in order to expand their knowledge of the big ideas of junior math and of how students learn mathematics. Initially, lead teachers will need time to internalize and test out some of their new learning about mathematics education. It would be unrealistic and unreasonable to expect that lead teachers would immediately be able to share new learning with other staff without having integrated such learning into their own instruction. This is particularly true for math lead teachers, who must now compensate for the lack of a systemic focus across Ontario in previous years on increasing content knowledge of mathematics and of mathematics pedagogy.

Math lead teachers will also need time to learn about effective facilitation skills and strategies for working with other staff. Since many elementary teachers have some level of math anxiety, math lead teachers will need to learn about developing ways of creating constructive learning environments when they are working with other staff members. Since numeracy initiatives are a shared responsibility of the entire school community, lead teachers should not be viewed as holding the sole responsibility for professional learning in the school.

After sufficient training and time, math lead teachers may be asked to:

- offer support to the junior team in the school;
- share resources and ideas with others on an ongoing basis;

- act as a mentor, when appropriate;
- model lessons for other teachers, when appropriate;
- be a source of support for other teachers;
- be a team leader for the junior mathematics teachers in the school;
- continue to attend professional development sessions;
- continue to implement new mathematical strategies in their own classroom.

We recognize that the role of lead teacher will vary from board to board as a reflection of the context and culture of the local school district. Lead teachers will need continued and ongoing support from principals, math facilitators, consultants/coordinators, and board leaders. They will benefit from opportunities to network and share with other lead teachers throughout the year as well as from continuing professional development that focuses on sound mathematical teaching and good pedagogy.

The choice of a lead teacher is crucial to the success of this initiative. Teachers do not need to have any background in the area but should be interested in learning more about mathematics. There should be different lead teachers for math and for literacy, because the demands on a teacher attempting to perform both functions would be too great and opportunities to develop and later share his or her knowledge and skills would be limited.

Role of the Ministry of Education

The Ministry of Education plays an integral role in the implementation of all of the components of the junior mathematics initiative. The ministry has taken a lead role in the development of a comprehensive plan to improve student achievement in mathematics and to provide teachers with some of the supports they need. The articulation of a common vision of mathematics instruction across Ontario will benefit all members of the educational community and will promote sharing, collaboration, and networking both within individual boards and among boards of education. At the same time, we encourage the ministry to be flexible in allowing boards to create the individualized implementation plans required to meet the specific and diverse needs of their teachers and students.

We recommend that the ministry continue to support mathematics education in Ontario by:

- ensuring alignment between projects and resources developed by different branches of the Ministry of Education, so that teachers are receiving consistent messages (e.g., alignment of the big ideas in the forthcoming *Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6* and in revisions to the Ontario curriculum);
- continuing to develop resources to support teachers in teaching mathematics;

- completing the big ideas for all strands of the curriculum and developing related instructional resources to ensure consistency across Ontario;
- providing adequate training for lead teachers to ensure that they have the knowledge and skills to perform the role effectively;
- continuing to base resources and training on current research about how children learn;
- providing access to training for all teachers in the junior division;
- ensuring that all school boards commit to a having mathematics as a system priority;
- recognizing that change in mathematics instruction will take time and that professional development and other supports must be sustained over an extended period of time to guarantee success.

Role of Faculties of Education

Faculties of education provide the foundation of teachers' formal education in mathematics instruction. If the recommended changes in instruction are to take place, prospective teachers will need greater support before entering into the profession than they now receive.

Faculties must devote sufficient time (a minimum of a half course or 36 hours) to the mathematics instruction of prospective teachers. In recognition of their important role in this area, many Ontario faculties have already moved to devoting more course hours to mathematics instruction than has been the case in the past and have begun offering extra non-credit classes or tutoring to strength students' mathematics knowledge.

6 Conclusion

We know that the mathematical demands on our students will only increase as they progress through school, take up their adult lives at home and in the workplace, and become active members of their society. Mathematical literacy will be essential to their future success in these areas. In order to function in a mathematically literate way in the future, students must have a strong foundation in mathematics in their elementary school years. As illustrated throughout this report, a strong foundation involves much more than the rote application of procedural knowledge. All students should be able to:

- understand, make sense of, and apply mathematics;
- make connections between concepts and see patterns throughout mathematics;
- communicate their reasoning and, equally, listen to others reason mathematically;
- develop the capacity and the flexibility of thinking that will allow them to tackle new areas of mathematics and new problems;
- enjoy mathematics and be willing to persevere in doing mathematics;
- view themselves as capable of doing mathematics.

We have discussed in detail the types of instructional practice that will most effectively lay the foundation of strong mathematical literacy. The ideas that we have proposed build on the work under way in the primary and junior divisions of Ontario schools. The goal of this document is to broaden and deepen the work already begun by providing a concrete vision of effective mathematics instruction at the junior level. Such instruction:

- is based on problem solving and inquiry;
- explores worthwhile and interesting mathematical tasks;
- begins with and capitalizes on students' thinking;
- develops a genuine mathematical community in the classroom;
- includes varied and relevant instructional and assessment strategies;
- is taught by teachers who have a strong knowledge of mathematical content and pedagogy.

To make this vision a reality, educators need to have opportunities for ongoing, sustained, quality professional development in mathematics. They also need sufficient support and resources in the classroom.

We recognize that time and effort will be needed to make the ideas discussed in this document bear fruit. The project is an ambitious one. We believe, however, that the journey will be worthwhile; indeed, is essential for our students' future capacity. We call upon all involved, including parents, teachers, principals and boards, math facilitators, board and district administrators, institutions of higher learning, and professional organizations, to focus on making this vision a reality.

Appendix A: A List of Recommended Manipulatives

Manipulative	Recommended for Every Classroom	Recommended for Every Grade/Division
Abacus	X	
Attribute blocks	X	
Balance and weights	X	
Base 10 materials, with transparent overhead set	X	
Calculators with two-line display	X	
Overhead calculator	X	
Centicubes	X	
Connecting plastic shapes to build 2-D shapes and 3-D nets		X
Coloured tiles		X
Coloured relational rods		X
Counters	X	
Dice/Numbered cubes	X	
Fraction kit	X	
Geoboards – transparent (5 x 5 and 11 x 11)	X	
Geometric solids	X	
Graduated beakers		X
Ninety-nine chart, Hundreds chart, hundreds board	X	
Measuring spoons		X

Manipulative	Recommended for Every Classroom	Recommended for Every Grade/Division
Measuring cups		X
Measuring tapes		X
Metre stick	X	
Mirror	X	
Plastic transparent tools		X
Playing cards	X	
Money	X	
Number lines	X	
Pattern blocks, with transparent overhead set	X	
Pentominoes		X
Plastic polygons (wide variety of triangles and regular and irregular quadrilaterals)	X	
Protractors	X	
Rekenrek (Dutch calculating frame)		X
Safety compass		X
Scales	X	
Spinners (number, colour)	X	
Square flat tiles	X	
Standard masses	X	
Stamps of various mathematical manipulatives (e.g., pattern blocks, tangrams, base 10 materials)		X

Manipulative	Recommended for Every Classroom	Recommended for Every Grade/Division
Stopwatch		X
Tangrams, with transparent overhead set		X
Thermometers		X
Two-colour counters	X	
3-D solids	X	
Trundle wheel		X

Appendix B: Professional Resources for Teachers

The following is a list of selected resource books that teachers can use to support their instruction. We encourage school boards and individual schools to continue to develop and maintain a complete and up-to-date collection of these resources.

English Language Resources

Overall Instruction Guides

Burns, M. (2000). *About teaching mathematics: A K–8 resource* (2nd ed.). Sausalito, CA: Math Solutions Publications.

Carpenter, T. P., Loef Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Chapin, S., & Johnson, A. (2000). *Math matters: Understanding the math that you teach.* Sausalito, CA: Math Solutions Publications.

Haylock, D., & McDougall, D. (1999). *Mathematics every elementary teacher should know*. Trifolium Books.

Lester, K., & Charles, R. I. (Eds.). (2003). *Teaching mathematics through problem solving: Prekindergarten–Grade 6*. Reston, VA: National Council of Teachers of Mathematics.

Sullivan, P., & Lilburn, P. (1997). *Good questions for math teaching: Why ask them and what to ask*. Sausalito, CA: Math Solutions Publications.

Van de Walle, J., & Folk, S. (2005). *Elementary and middle school mathematics: Teaching developmentally* (Canadian edition). New York: Longman.

Specific Content Area Resources and Units

Burns, M. (1989). *A collection of math lessons from Grades 3 through 6*. New York: Math Solutions Publications.

Burns, M. (1995). *Math by all means: Probability, Grades 3–4*. New York: Math Solutions Publications.

Burns, M. (1996). *50 problem-solving lessons*. New York: Math Solutions Publications.

Burns, M. (2001). *Lessons for introducing fractions*. Sausolito, CA: Math Solutions Publications.

Burns, M., & Humphries, C. (1990). *A collection of math lessons from Grades 6 through 8*. New York: Math Solutions Publications.

Erickson, T. (1989). *Get it together*. Berkeley, CA: EQUALS.

ETA/Cuisenaire Publications. Supersource series, Grades 3–4 and 5–6. Urbana, IL: Author.

National Council of Teachers of Mathematics. (2001). Navigations series, Grades 3–5. Reston, VA: Author.

Ohanian, S., & Burns, M. (1997). *Math by all means: Division, Grades 3–4*. Sausalito, CA: Math Solutions Publications.

Rectanus, C. (1994). *Math by all means: Geometry, Grades 3–4*. Sausalito, CA: Math Solutions Publications.

Rectanus, C. (1997). *Math by all means: Area and perimeter, Grades 5–6*. Sausalito, CA: Math Solutions Publications.

Skinner, P. (1998). *It all adds up*. Sausalito, CA: Math Solutions Publications.

Wickett, M., & Burns, M. (2001). *Lessons for extending multiplication*. Sausolito, CA: Math Solutions Publications.

Wickett, M., Kharas, K., & Burns, M. (2002). *Lessons for algebraic thinking: Grades 3–5*. Sausalito, CA: Math Solutions Publications.

Specific Content Area Theory

Fosnot, C. T., & Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Portsmouth, NH: Heinemann.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Erlbaum.

Schifter, D. (1996). *What's happening in math class?* (Vol. 1). New York: Teachers College Press.

Schifter, D., Bastable, V., & Russell, S. J. (1999). *Developing mathematical ideas: Number and operations, part 1– Building a system of tens*. Parsippany, NJ: Dale Seymour Publications.

Schifter, D., Bastable, V., & Russell, S. J. (1999). *Developing mathematical ideas: Number and operations, part 2 – Making meaning for operations*.

Children's Literature and Resources Books

Bresser, R. (1995). *Math and literature: Grades 4–6*. New York: Math Solutions Publications.

Thiessen, D. (Ed.). (2004). *Exploring mathematics through literature: Articles and lessons for Prekindergarten through Grade 8*. Reston, VA: National Council of Teachers of Mathematics.

Whitin, D. J., & Wilde, S. (1995). *It's the story that counts: More children's books for mathematical learning, K–6*. Portsmouth, NH: Heinemann.

Assessment

National Council of Teachers of Mathematics. (2001). *Mathematics assessment: Cases and discussion questions for Grades K–5*. Reston, VA: Author.

Stenmark, J., & Bush, W. (Eds.). (2003). *Mathematics assessment: A practical handbook for Grades 3–5*. Reston, VA: National Council of Teachers of Mathematics.

Special Needs

Sliva, J. (2004). *Teaching inclusive mathematics to special learners, K–6*. Thousand Oaks, CA: Corwin Press.

New Teachers

Burns, M., & Silbey, R. (1999). *So you have to teach math? Sound advice for K–6 Teachers*. White Plains, NY: Math Solutions Publications.

Elementary Teachers' Federation of Ontario. (2004). *Making math happen in the junior years*. Toronto: Author

Leadership Resources

Burns, M. (1999) *Leading the way*. Sausalito, CA: Math Solutions Publications.

Love, N. (2002). *Using data/getting results: A practical guide for school improvement in mathematics and science*. Norwood, MA: Christopher-Gordon (on behalf of TERC).

Miles Grant, C., Davidson, E., Shulman Weinberg, A., Scott Nelson, B., Sassi, A., & Bleiman, J. *Lenses on learning: Instructional leadership in mathematics (modules 1, 2, and 3)*. Parsippany, NJ: Dale Seymour Publications.

Mirra, A. J. (2003). *Administrator's guide: How to support and improve mathematics education in your school*. Reston, VA: National Council of Teachers of Mathematics.

National Association of Elementary School Principals. (2002). *What principals need to know about teaching math*. Alexandria, VA: Author.

Working With Parents

Litton, N. (1998). *Getting your math message out to parents: A K–6 resource*. Sausalito, CA: Math Solutions Publications.

Thompson, V., & Mayfield-Ingram, K. (1998). *Family math for the middle years*. Berkeley: University of California.

Communication

Burns, M. (1995). *Writing in math class: A resource for Grades 2–8*. Sausalito, CA: Math Solutions Publications.

Chapin, S., O'Connor, C., & Anderson, N. (2003). *Classroom discussions: Using math talk to help students learn, Grades 1–6*. Sausalito, CA: Math Solutions Publications.

Whitin, P., & Whitin, D. J., (2000). *Math is language too*. Urbana, IL: National Council of Teachers of English and National Council of Teachers of Mathematics.

French-Language Resources

Ressources professionnelles

Centre franco-ontarien de ressources pédagogiques. (2001). *Les mathématiques… un peu, beaucoup, à la folie! Géométrie (4e à la 6e année)*. Ottawa: Author.

Centre franco-ontarien de ressources pédagogiques. (2001). *Modélisation et algèbre (4e à la 6e année)*. Ottawa: Author.

Centre franco-ontarien de ressources pédagogiques. (2002). *Recueil de pratiques réussies en mathématiques de la 6e à la 9e année*. Ottawa: Author.

Centre franco-ontarien de ressources pédagogiques. (2003). *Recueil de pratiques réussies en mathématiques de la 1re à la 5e année*. Ottawa: Author.

Centre franco-ontarien de ressources pédagogiques. (2004). *Traitement des données et probabilité (4e à la 6e année)*. Ottawa: Author.

Lemoyne, G., & Conne, F. (1999). *Le cognitif en didactique des mathématiques*. Montreal: Les Presses de l'Université de Montréal.

Cmathématique website. www.cmathématique.com

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Baek, J. M. (1998). Children's invented algorithms for multidigit multiplication problems. In L. Morrow (Ed.), *Teaching and learning of algorithms in school mathematics* (pp. 151–160). Reston, VA: National Council of Teachers of Mathematics.

Ball, D., & Cohen, D. (1996). Reform by the book: What is – or might be – the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, *25*(9), 6–8.

Baroody, A., & Ginsburg, H. (1990). Children's learning: A cognitive view. *Journal for Research in Mathematics.* Monograph 4.

Battista, M. (1999). The mathematical miseducation of America's youth. *Phi Delta Kappan, 80*(6), 425–433.

Battista, M. (2003). Understanding students' thinking about area and volume measurement. In D. H. Clements and G. W. Bright (Eds.), *Learning and teaching measurement* (pp. 122–142). Reston, VA: National Council of Teachers of Mathematics.

Bednarz, N. (2000). Formation continue des enseignants en mathématiques : Une nécessaire prise en compte du contexte. In P. Blouin and L. Gattuso (Eds.), *Didactique des mathématiques et formation des enseignants* (pp. 61–78). Mont-Royal, QC: Modulo Éditeur.

Birch, S. H., & Ladd, G. W. (1997). The teacher-child relationship and children's early school adjustment. *Journal of School Psychology, 35*, 61–79.

Black, P., & Wiliam, D. (1998, October). Inside the black box. *Phi Delta Kappan, 80*(2), 139–148.

Boaler, J. (2002). Learning from teaching: Exploring the relationship between "reform" curriculum and equity. *Journal for Research in Mathematics Education, 33*(4), 239–258.

Bresser, R. (1995). *Math and literature: Grades 4–6*. New York: Math Solutions Publications.

Bresser, R. (2003, February). Helping English-language learners develop computation fluency. *Teaching Children Mathematics, 9*(6), 294.

Burch, P., & Spillane, J. P. (2001). *Elementary school leadership strategies and subject matter: The cases of mathematics and literacy instruction*. Paper presented at the American Educational Research Association Meetings, Seattle, WA.

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Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. New York: Kluwer.

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