Lifeguard Procedures Report LEVEL 4

A

LIFEGUARD PROCEDURES

Assumptions

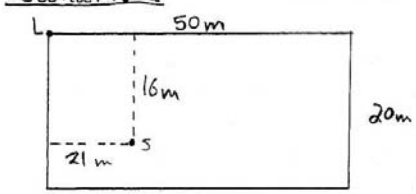
Before we address this problem, we will state the assumptions we are relying on as we come up with the solution. They are:

- The running speed of the lifeguard is always faster than the swimming speed
- The lifeguard is always located at the same corner of the pool
- The pool is rectangular in shape
- The pool is indoors, so factors such as wind or rain do not come into play
- The pool deck is suitable for fast running
- Factors such as the depth of the swimmer and the depth of the lifeguard's dive are negligible.

Possible Scenarios

We will now use different scenarios to illustrate the problem and its possible solutions.

Scenario 1

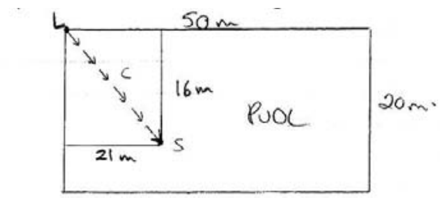


In this scenario, the swimmer is located 16 m from the top of the pool and 21 m from the side. The lifeguard, in his normal position, has a few different options… For now, we will assume a running speed of 5 m/s and a swimming speed of 3 m/s.

We will consider three options. My hypothesis is that option 3 will take the least time.

B

Option 1 – The lifeguard swims directly to the swimmer



L = lifeguard position
S = swimmer position

The path is represented by the dashed arrows.
To calculate the time required for this we must first find the distance.
Using Pythagorean theorem: we can see a right triangle here.

$c^2 = a^2 + b^2$, $a = 21$, $b = 16$, solving for c.

$c^2 = 21^2 + 16^2$

$c = \sqrt{697}$

$c = 26.4$ m

∴ the distance swam is 26.4 m.

To calculate time, we use the speed formula:

$\text{speed} = \frac{\text{distance}}{\text{time}}$, solving for time

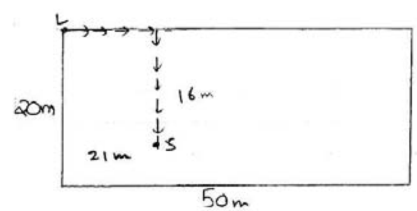
$(t)\ \text{time} = \frac{\text{distance}\ (d)}{\text{speed}\ (s)}$ where $d = 26.4$ m, $s = 3$ m/s

$t = \frac{26.4\text{ m}}{3\text{ m/s}}$

$t = 8.8$ sec.

∴ the time taken for the rescue is 8.8 seconds.

Option 2 – The lifeguard runs along the deck to a point perpendicular to the swimmer, then swims to the swimmer.



L = lifeguard position
S = swimmer position

C

The path of the lifeguard is represented by the arrows.
To find the time required for this approach we can split up each segment of it.
For the running portion:

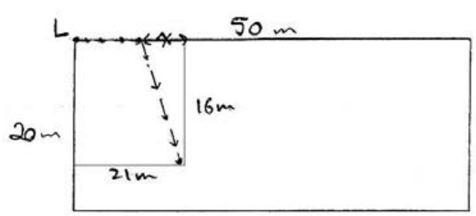
$t = \frac{d}{s}$, $d = 21\,m$, $s = 5\,m/s$
$t = \frac{21}{5}$
$t = 4.2$ seconds

For the swimming portion:

$t = \frac{d}{s}$, $d = 16\,m$, $s = 3\,m/s$
$t = \frac{16}{3}$
$t = 5.3\overline{3}$ seconds

Adding these two together gives us 9.53 seconds, which is a longer time than that of the first approach. This suggests that swimming directly to the swimmer is quicker than running along the deck and swimming perpendicularly toward the swimmer.

Option 3 – Variable Run & Swim Distances
With this approach, the lifeguard runs along the deck and enters the pool at an angle toward the swimmer. The distance run by the lifeguard is variable.



In this diagram, x represents a variable distance that the lifeguard does not run.

D

We can derive an equation for the time of this approach in terms of x.

$t =$ run time + swim time
$t = \frac{21 - x}{5} + \frac{\sqrt{256 + x^2}}{3}$

To find the value of x that will give us a minimum time, we can use calculus. Finding the derivative, t′ and setting it to zero and solving for x will give us the value of x that will minimize time.

$t' = -\frac{1}{5} + \frac{\frac{1}{2\sqrt{256+x^2}} \cdot 2x \cdot 3 - 0}{9}$

$t' = -\frac{1}{5} + \frac{\frac{3x}{\sqrt{256+x^2}}}{9}$

$t' = \frac{\frac{x}{\sqrt{256+x^2}}}{3} - \frac{1}{5}$

$t' = \frac{x}{3\sqrt{256+x^2}} - \frac{1}{5}$, setting $t' = 0$

$0 = \frac{x}{3\sqrt{256+x^2}} - \frac{1}{5}$

$\frac{1}{5} = \frac{x}{3\sqrt{256+x^2}}$

$3\sqrt{256+x^2} = 5x$, squaring both sides

$9(256 + x^2) = 25x^2$
$2304 + 9x^2 = 25x^2$
$2304 = 16x^2$
$144 = x^2$
$12 = x$

LEVEL 4

E

If x = 12, this means that the lifeguard runs 9 m and then enters the pool and swims to the swimmer.

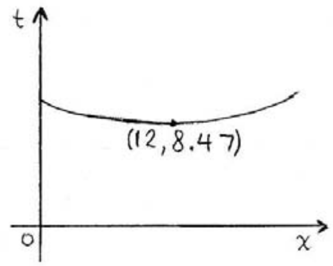
The time taken for $x = 12$, is

$$t = \frac{21-12}{5} + \frac{\sqrt{256+12^2}}{3}$$

$$t = \frac{9}{5} + \frac{\sqrt{400}}{3}$$

$$t = \frac{9}{5} + \frac{20}{3}$$

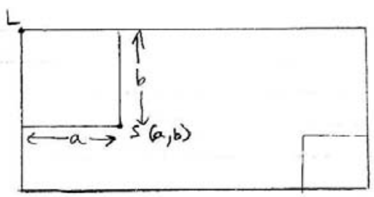
$t = 8.47$ seconds



This time is clearly lower than that of the first 2 options, so this lifesaving approach is the best.

Generic

The above scenario relied on set values for the distances. Since it is unlikely that a swimmer in need of help will always be in the same position, we will use the formula from above to calculate a ratio.



L = lifeguard
S = swimmer

Let a represent the horizontal distance from lifeguard to swimmer.
Let b represent the vertical distance from lifeguard to swimmer.
These distances could be considered either way, so we will let a be the longer distance: a > b

We will make it clear that the lifeguard <u>will always run along the side which has the greater distance</u> – in other words, L will run along "a".

F

In our earlier examples we also used constants for speeds. In this section, we will let r be the running speed and s be the swimming speed.
With r, s, a, and b, the equation for t looks like this:

$$t = \frac{a-x}{r} + \frac{\sqrt{b^2+x^2}}{s}$$

If we want to minimize time, differentiate.

$$t' = -\frac{1}{r} + \frac{\frac{1}{2\sqrt{b^2+x^2}} \cdot \cancel{2}x \cdot s - 0}{s^2}$$

$$t' = -\frac{1}{r} + \frac{\frac{\cancel{s}x}{\sqrt{b^2+x^2}}}{s^{\cancel{2}}}$$

$$t' = \frac{x}{s\sqrt{b^2+x^2}} - \frac{1}{r}, \text{ when } t' = 0$$

$$0 = \frac{x}{s\sqrt{b^2+x^2}} - \frac{1}{r}$$

$$\frac{1}{r} = \frac{x}{s\sqrt{b^2+x^2}}$$

$$s\sqrt{b^2+x^2} = xr$$

$$s^2(b^2+x^2) = x^2r^2$$

$$s^2b^2 + s^2x^2 = x^2r^2$$

$$s^2b^2 = x^2r^2 - s^2x^2$$

$$s^2b^2 = x^2(r^2-s^2)$$

G

$$\frac{s^2b^2}{r^2-s^2} = x^2$$

$$\sqrt{\frac{s^2b^2}{r^2-s^2}} = x$$

$$\frac{sb}{\sqrt{r^2-s^2}} = x$$

The denominator, $\sqrt{r^2-s^2}$, suggests that $r^2 \geq s^2$, or $r \geq s$ since they are both positive. We already stated in our assumptions that the running speed is greater than the swim speed, so $r > s$.

I suggest that when a lifeguard is hired, he or she should measure their swimming and running speeds. If these are known, an easy way to save people can be found.

For example, let's say the running speed is 5 m/s and swim speed is 3 m/s. Then: to minimize time

$$\frac{3b}{\sqrt{5^2-3^2}} = x$$

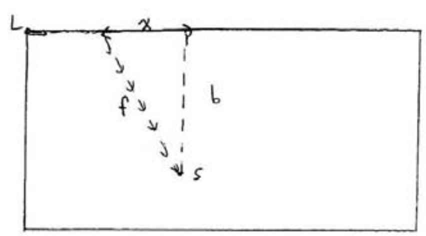
$$\frac{3b}{\sqrt{25-9}} = x$$

$$\frac{3b}{\sqrt{16}} = x$$

$$\frac{3}{4}b = x$$

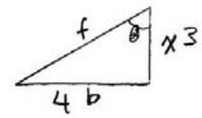
in this case, we have a simple ratio. This ratio will help us in deciding our x-value.

Observe the following diagram:



H

We have a right triangle in sides x, b, and f.



We know the ratio of x to b, no matter what the values, is 3:4.

Using the tangent function to determine angle θ

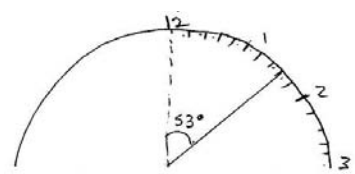
$$\tan = \frac{opp}{adj}$$

$$\tan\theta = \frac{4}{3}$$

$$\theta = \tan^{-1}\frac{4}{3}$$

$$\theta = 53°$$

θ = 53° If the lifeguard had measured his running and swimming speeds, he could easily use the above set of calculations to calculate this "critical angle" as I like to call it. How does this help?



In a clock, each minute represents 6°. For this particular lifeguard, 53° would be about halfway between 1 o'clock and 2 o'clock.

This is what I recommend the lifeguard does:
When he sees the swimmer in trouble, he should begin running along the side of the deck <u>which has the longer distance</u> ("a", if you remember). As he runs he looks straight ahead and visualizes half of a clock in his view, with 12 o'clock being straight ahead and 3 o'clock to the right. When the swimmer appears in the corner of his eye between 1 and 2 o'clock, he jumps in and swims toward the swimmer. This minimizes the time taken, as we have seen from the calculus in the report.

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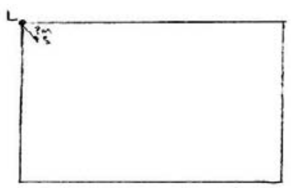
I

I suggest that every lifeguard, when hired, measures his run speed and swim speed. following this, he calculates the ratio of x to b using the formula

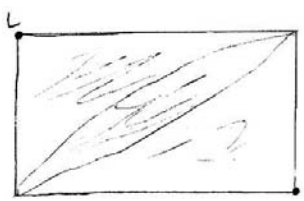
$$\frac{sb}{\sqrt{r^2 - s^2}} = x$$

With this ratio and the tangent function, the lifeguard can calculate his or her “critical angle” as we saw above. This takes all of five minutes, but can mean the difference between life and death.

I want to stress that this method should only be used when the swimmer is a sizeable distance, DIAGONALLY, from the lifeguard. For example:



With the swimmer this close to the lifeguard, it is more effective to just jump in. The time taken to get down and begin running would hurt the process.

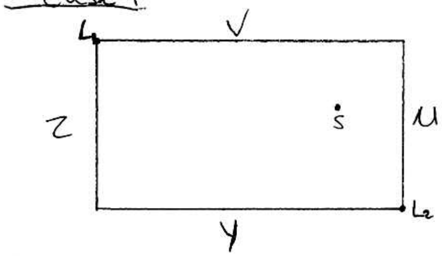


I also recommend that two lifeguards be positioned at opposite corners of the pool. This ensures that a lifeguard will be close to an injured swimmer. The lifeguard that is closer to the swimmer should take the job.

J

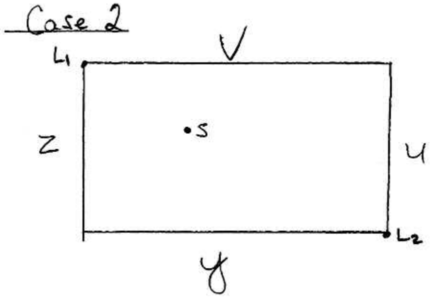
Now I will show a few case scenarios:

Case 1



In this scenario, L_2 will run along side u (because the vertical distance to the swimmer is greater), then dive in at the critical angle and swim to S.

Case 2



In this case, L_1 is closer. He should run along side V (the horizontal distance being the greater) then jump in when the swimmer appears in the right clock position to save him.

Teacher's Notes

Knowledge and Understanding

- The student uses a mathematical model with a high degree of effectiveness. He or she lists several reasonable simplifying assumptions, but there are omissions (e.g., the assumption that the path to the swimmer is not blocked by other swimmers). The student forms the derivative accurately, including a derivative for the general case. However, in using the quotient rule, he or she does not choose the most efficient method. The student solves the resulting equations accurately and efficiently.

Thinking

- The student interprets the solutions to the equations with a high degree of effectiveness. By substituting the speeds from Appendix A into the general formula for x, he or she clearly demonstrates that $\frac{3}{4}b = x$ for these speeds, and that the "critical angle" between the running and swimming directions is 53°.
- The student formulates and tests hypotheses with a high degree of effectiveness. He or she states that "My hypothesis is that option 3 will take the least time". The student confirms this hypothesis for scenario 1. He or she then considers all scenarios by writing the derivative for the general case and solving the resulting equation, with distances and speeds as variables.

Communication

- The student communicates information in diagrams or graphs with a high degree of clarity. Diagrams of the various options are accurate and clearly labelled. The recommendations include an effective diagram of half a clock face as an aid in angle estimation. The graph shows the results for scenario 1. The student includes the coordinates of the minimum on the graph, but does not show the endpoints and their coordinates.
- The student integrates text and mathematical forms with a high degree of effectiveness. The report flows well, and the student includes clear and detailed explanations to show his or her thinking. Most of the report is organized into logical sections, but it lacks an introduction. The addition of a "Recommendations" subhead would more clearly indicate where the recommendations begin.

Application

- The student uses formulas with a high degree of effectiveness. He or she chooses appropriate formulas when considering the three options, and correctly writes general formulas for t and x for option 3. The student substitutes appropriate values into the formulas, i.e., distance and speed values from Appendix A. The student correctly uses a trigonometric formula to find the angle between the running and swimming directions.
- The student recommends and justifies a course of action with a high degree of effectiveness. The student recommends that "when a lifeguard is hired, he or she should measure their swimming and running speeds" to find the appropriate angle between the running and swimming directions. The student suggests that the lifeguard should visualize "half of a clock" as an aid in estimating the angle during the rescue. However, maintaining an awareness of the swimmer, while looking "straight ahead" and running along the side of the pool, might be difficult in practice. The student further recommends that there be two lifeguards, and describes which side of the pool a lifeguard should run along to make a rescue.

LEVEL 4

Comments

This work is representative of a solid level-4 performance. The student demonstrates a high degree of achievement of the expectations in all four categories of knowledge and skills.

Next Steps

In order to improve his or her performance, the student needs to:

- include more simplifying assumptions;
- form the derivative more efficiently;
- label the graph fully;
- add an introduction to the report;
- consider the ease of implementation of the recommendations.